OPTION PRICING UNDER THRESHOLD AUTOREGRESSIVE MODELS BY THRESHOLD ESSCHER TRANSFORM

TAK KUEN SIU
Department of Actuarial Mathematics and Statistics
Heriot-Watt University, Edinburgh, United Kingdom

HOWELL TONG
London School of Economics, United Kingdom

HAILIANG YANG
Department of Statistics and Actuarial Science
The University of Hong Kong, Pokfulam Road, Hong Kong

Abstract. This paper develops a valuation model for options under the class of self-exciting threshold autoregressive (SETAR) models and their variants for the price dynamics of the underlying asset using the self-exciting threshold autoregressive Esscher transform (SETARET). In particular, we focus on the first generation SETAR models first proposed by Tong (1977, 1978) and later developed in Tong (1980, 1983) and Tong and Lim (1980), and the second generation models, including the SETAR-GARCH model proposed in Tong (1990) and the double-threshold autoregressive heteroskedastic time series model (DTARCH) proposed by Li and Li (1996). The class of SETAR-GARCH models has the advantage of modelling the non-linearity of the conditional first moment and the varying conditional second moment of the financial time series. We adopt the SETARET to identify an equivalent martingale measure for option valuation in the incomplete market described by the discrete-time SETAR models. We are able to justify our choice of probability measure by the SETARET by considering the self-exciting threshold dynamic utility maximization. Simulation studies will be conducted to investigate the impacts of the threshold effect in the conditional mean described by the first generation model and that in the conditional variance described by the second generation model on the qualitative behaviors of the option prices as the strike price varies.

1. Introduction and Summary. The seminal work of Black and Scholes (1973) and Merton (1973) introduced a compact preference-free option pricing formula and laid the foundation of modern option pricing theory. The celebrated Black-Scholes formula does not involve an investor’s risk preference or subjective view and depends on some observable and estimable parameters. Due to its compact form and computational simplicity, the Black-Scholes formula enjoys great popularity in the finance industries. One of the important economic insights underlying the preference-free option-pricing result is the concept of perfect replication of contingent claims by continuously adjusting a self-financing portfolio. Harrison and Kreps (1979), Harrison and Pliska (1981, 1983) established a solid mathematical
foundation for the relationship between the no-arbitrage principle and the notion of risk-neutral valuation using the modern language of probability theory. They showed that the absence of arbitrage opportunities is equivalent to the existence of an equivalent martingale measure. If the securities market is complete, there is a unique martingale measure and hence a unique price of any contingent claim. If the market is incomplete, there is more than one equivalent martingale measure and hence a range of no-arbitrage prices for a contingent claim.

Different approaches have been proposed in the literature to determine an equivalent martingale measure for option valuation in an incomplete market. Föllmer and Sondermann (1986), Föllmer and Schweizer (1991) and Schweizer (1996) identified a unique equivalent martingale measure by minimizing the variance of the hedging loss. Davis (1997) adopted the marginal rate of substitution and determined an equivalent martingale pricing measure by a utility maximization problem. Gerber and Shiu (1994) provided an elegant way to choose an equivalent martingale measure using the Esscher transform, a time-honored tool in actuarial science introduced by Esscher (1932). Their approach is flexible enough to incorporate various parametric models within the class of infinitely divisible distributions. The option price by the Esscher transform can be justified by maximizing the expected power utility of an economic agent. The seminal work by Gerber and Shiu (1994) provides an important insight in bridging the gap between the financial and insurance pricing problems in an incomplete market. Bühlmann et al. (1996) generalized the classical notion of Esscher transform to stochastic processes. They introduced the conditional Esscher transform in order to incorporate a richer theory of semi-martingale under the no-arbitrage condition in the context of Gerber-Shiu's option-pricing model. Siu, Tong and Yang (2004) adopted the conditional Esscher transform in Bühlmann et al. (1996) to determine an equivalent martingale measure for option valuation under the class of GARCH models with innovations having an infinitely divisible distribution.

Recently, there is much interest in the interaction between financial econometrics and option pricing theory. Much of the literature focuses on the option valuation problem under conditional heteroskedastic models for the underlying risky assets, like the ARCH-type models. Earlier work in this area includes Duan (1995), which is the first one to provide a solid theoretical foundation for option valuation when the price dynamics of the underlying asset are governed by a GARCH model. Since then, the option pricing problem under the GARCH models has been studied extensively in the literature. There is relatively little work concerning the option valuation problem when the price dynamics of the underlying asset are governed by other non-linear time series models, such as the class of threshold autoregressive models. The class of self-exciting threshold autoregressive (SETAR) models was first proposed by Tong (1977, 1978) and later developed in Tong (1980, 1983) and Tong and Lim (1980). It has been widely adopted in various fields, such as economics and finance, actuarial science, medical and biological sciences, engineering sciences, ecology and environment sciences, etc. See Tong (1990) for examples from diverse fields. The main idea of the SETAR models is to use a piecewise linear approximation as a proxy for the general non-linear autoregressive time series model. The monograph

\[1\] Siu et al. (2001) provided another direction of generalizing the classical notion of Esscher transform, namely, the random Esscher transform, by assuming the Esscher parameter as a random variable. One application of the random Esscher transform is to generate a family of random generalized “scenarios” for risk measurement.
by Tong (1990) provides a comprehensive discussion on the SETAR models and other important non-linear time series models.

In the fields of economics and finance, Heish (1991) adopted the SETAR models to investigate the non-linear dynamics of the financial returns and chaos in financial markets. Yadav, Pope and Paudyal (1994) adopted the threshold autoregressive models to explain many empirical phenomena in futures markets. Clements and Smith (2001) provided simulation studies on the forecasting performance of the empirical SETAR models for the exchange-rate time series. Chan, Wong and Tong (2004) investigated the use of the class of SETAR models and its variants in pricing insurance products, analyzing the results of experience studies and validating actuarial assumptions. The class of SETAR models is also useful to incorporate the regime-switching phenomenon of financial returns. The regime-switching phenomenon is induced by the values of the process. It has been mentioned in Krager and Kugler (1993) and Franses and Dijk (2000) that currency exchange rates may exhibit regime-switching behavior due to the intervention of monetary authorities to stabilize the currency during the period of either large appreciation or large depreciation of currency rates. In this case, the regime depends on the change in the level of the currency exchange rates.

In this paper, we develop a valuation model for options under the class of self-exciting threshold autoregressive (SETAR) models and their variants for the price dynamics of the underlying asset using the self-exciting threshold autoregressive Escher transform (SETARET). In particular, we focus on the first generation SETAR models and the second generation models, including the SETAR-GARCH models proposed in Tong (1990) and the double-threshold autoregressive heteroskedastic time series model proposed by Li and Li (1996). The class of SETAR-GARCH models has the advantage of modelling the non-linearity of the conditional first moment and the varying conditional second moment of the financial time series. We consider the general case by assuming that the innovations have an infinitely divisible distribution. Due to the fact that the market described by the SETAR model is incomplete, there are infinitely many equivalent martingale measures for option valuation. Here, we adopt the SETARET to identify an equivalent martingale measure for option valuation in the incomplete market. Then, we consider two parametric forms for the conditional distributions of the innovations, namely, the normal distribution and the shifted-gamma distribution. We are able to justify our choice of probability measure by the SETARET by considering a self-exciting threshold dynamic utility maximization. We will conduct simulation studies to investigate the impacts of the threshold effect in the conditional mean described by the first generation model and that in the conditional variance described by the second generation model on the qualitative behaviors of the option prices as the strike price varies. This paper is organized as follows.

The next section presents the main idea of our model. We consider the first generation models described in Tong (1990), namely, the class of the SETAR model and develop the SETARET for determining an equivalent martingale measure for option valuation. To provide a more general result, we consider the class of the SETAR model with innovations having an infinitely divisible distribution. The option price is then justified by the self-exciting threshold dynamic utility maximization. Section 3 considers the second generation models, namely, the SETAR-GARCH models, and the double-threshold autoregressive heteroskedastic time series (DTARCH) model. As in Section 2, we consider the case with innovations having an infinitely divisible
distribution. The SETARET can also be used to determine an equivalent martingale pricing measure for option valuation in the context of the second generation models. Section 4 considers some parametric cases of the first generation models and the second generation models with innovations having normal distributions and shifted gamma distributions. We also deal with different special cases of the first generation models and the second generation models, such as the discrete-time linear autoregressive (AR) models, the ARCH models and the AR-ARCH models. Simulation results for the option prices implied by different models will be presented and discussed in Section 5. The final section suggests some possible topics for further investigation.

2. Option Pricing Under First Generation Models. In this section, we deal with the option pricing problem when the dynamics of the underlying risky asset are governed by a SETAR model. We develop a modified version of the conditional Esscher transform, namely, the self-exciting threshold autoregressive Esscher transform (SETARET), which is tailor-made for the option pricing problem under the class of threshold autoregressive time series models. In particular, we adopt the SETARET to determine an equivalent martingale measure for option valuation. The main advantage of using the SETARET is that it is flexible enough to incorporate various parametric distributional assumptions for the innovations of the SETAR models. In fact, we only assume that the distribution of the innovations process is infinitely divisible and that the moment generating function of the infinitely divisible distribution exists. The condition of finite moment generating function is a reasonable one since, otherwise, the conditional expectation of the one-period simple returns for the underlying asset is unbounded. To simplify our discussion, we consider a discrete-time financial model consisting of one risk-free bond $B$ and one risky asset $S$. In the sequel, we present the setup and the main idea of our model.

First, we fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is a real-world probability measure. Write $T$ for the time index set $\{0, 1, 2, \ldots \}$ of the discrete-time financial model. We equip the sample space $(\Omega, \mathcal{F})$ with the information structure $\Phi := \{\Phi_t\}_{t \in T}$. That is, for each $t \in T$, $\Phi_t$ represents the information set of all market information up to and including time $t$. Let $\{\xi_t\}_{t \in T}$ denote a stochastic process on $(\Omega, \mathcal{F})$ taking values on the real line $\mathbb{R}$, with $\xi_0 = 0$. For each $t \in T$, $\xi_t$ represents the innovation or the random noise of the underlying asset $S$. We assume that $\xi_t$ is known given $\Phi_t$ and that $\{\xi_t\}_{t \in T}$ is a sequence of independent and identically distributed (i.i.d.) random variables having an infinitely divisible distribution $F(\cdot|0, 1)$ with mean zero and variance one.

The SETAR model provides a piecewise linear approximation and parametric specification of some underlying non-linear autoregressive dynamics governing the logarithmic returns of the risky asset under consideration; we allow the innovations to have an infinitely divisible distribution in order to provide additional flexibility. First, let $r$ be the constant continuously compounded risk-free interest rate of the bond $B$ and $\lambda$ the constant unit risk premium representing a preference parameter. For each $t \in T$, we write $B_t$ and $S_t$ for the prices of the bond $B$ and the risky asset $S$ at time $t$, respectively. Let $Y_t$ denote the continuously compounded one-period rate of return $\ln(S_t/S_{t-1})$ of $S$. Then, we assume that, under $\mathbb{P}$, the dynamics of the logarithmic returns $\{Y_t\}_{t \in T}$ from the risky asset $S$ satisfy the following $k$-regime
**SETAR**($d; p_1, p_2, \ldots, p_k$):

\[ Y_t = \sum_{i=1}^{k} \left( r + \lambda \sigma_i - \frac{1}{2} \sigma_i^2 + \sum_{j=1}^{p_i} \beta_j^{(i)} Y_{t-j} + \sigma_i \xi_t \right) I_{\{r_{i-1} < Y_{t-d} \leq r_i\}}, \quad t \in T \setminus \{0\}, \quad (2.1) \]

where $k$ is the number of regimes in the model; $d$ is the delay parameter, which is a positive integer; $p_i$ is the autoregressive order in the $i$th regime of the model ($i = 1, 2, \ldots, k$); the threshold parameters satisfy the constraint $-\infty = r_0 < r_1 < \cdots < r_k < \infty$; $I_A$ is the indicator function of the event $A$; $\sigma_i^2$ is the conditional variance of $Y_t$ given $\Phi_{t-1}$ in the $i$th regime of the model. Tong (1990) pointed out that the SETAR model can incorporate the effect of conditional heteroscedasticity. It provides a parametric way other than the ARCH-type models to describe the effect of conditional heteroskedasticity. In fact, the conditional variance of $Y_t$ given $\Phi_{t-1}$ depends on the past value of the process $Y$, say $Y_{t-d}$. When $r_{i-1} < Y_{t-d} \leq r_i$, the conditional variance of $Y_t$ given $\Phi_{t-1}$ is $\sigma_i^2$. If we assume homoscedasticity across the regimes, $\sigma_i^2 = \sigma_1^2 = \cdots = \sigma_k^2 = \sigma^2$, where $\sigma^2$ is the common conditional variance of $Y_t$. The index $i$ represents a state of the world or regime of the model. Within each regime, the dynamics of the logarithmic returns $Y_t$ are assumed to be a linear autoregressive time series model. Note that the regime of the model at each time $t$ depends on the observable past history of the logarithmic returns $\{Y_t\}_{t \in T}$. Here, the regime at each time $t$ is governed by the value of $Y_{t-d}$, hence the term a self-exciting threshold autoregressive model. (See, Tong and Lim, 1980.)

Like most of the non-linear time series models, the class of SETAR models is time-irreversible in general. That is, the probabilistic properties of the model are different when it is investigated backward through time. Most of the linear time series models with Gaussian innovations, for instance Gaussian AR(1) processes, are time-reversible. Chan, Ho and Tong (2005) provided an exposition of time-reversibility of multivariate time series models. We further assume that the bond price process $\{B_t\}_{t \in T}$ follows:

\[ B_t = B_{t_{-1}} e^r, \quad B_0 = 1, \quad t \in T \setminus \{0\}. \quad (2.2) \]

In the following, we construct the SETARET for the class of the SETAR models for logarithmic returns $\{Y_t\}_{t \in T \setminus \{0\}}$. To simplify our discussion, we consider the first-order SETAR(1; 1, 1, 1) model for the logarithmic returns $Y_t$ as follows:

\[ Y_t = \sum_{i=1}^{k} \left( r + \lambda \sigma_i - \frac{1}{2} \sigma_i^2 + \beta^{(i)} Y_{t-1} + \sigma_i \xi_t \right) I_{\{r_{i-1} < Y_{t-1} \leq r_i\}}, \quad t \in T \setminus \{0\}. \quad (2.3) \]

The same method can be applied to deal with a general $k$-regime SETAR($d; p_1, p_2, \ldots, p_k$).

For the $i$th regime at time $t$, let $\theta_t^{(i)}$ denote a random variable which is measurable with respect to $\Phi_{t-1}$, for each $t \in T \setminus \{0\}$. We define a self-exciting threshold autoregressive Esscher parameter (SETAREP) at time $t$ as a random variable $\Theta_t$ in the following way:

\[ \Theta_t := \sum_{i=1}^{k} \theta_t^{(i)} I_{\{r_{i-1} < Y_{t-d} \leq r_i\}}, \quad t \in T \setminus \{0\}. \quad (2.4) \]

where $\theta_t^{(i)}$ is the value of the SETAREP when the model is in the $i$th regime at time $t$. 

**OPTION PRICING UNDER THRESHOLD AUTOREGRESSIVE MODELS 181**
Here, we consider the case that \( d = 1 \). Then, \( \Theta_t \) is given by:

\[
\Theta_t := \sum_{i=1}^{k} \theta^{(i)}_t I_{\{r_{i-1} < Y_{t-1} \leq r_i\}}, \quad t \in T \setminus \{0\}. \tag{2.5}
\]

Let \( M_{Y_t|\Phi_{t-1}}(z) \) be the moment generating function of the conditional distribution \( Y_t \) given \( \Phi_{t-1} \) under the real-world probability measure \( P \), where \( z \in \mathbb{R} \). That is,

\[
M_{Y_t|\Phi_{t-1}}(z) := \mathbb{E}_P(e^{zY_t}|\Phi_{t-1}) = \sum_{i=1}^{k} \int_{-\infty}^{\infty} e^{z y} dF(y|\Phi_{t-1}) = \sum_{i=1}^{k} \int_{-\infty}^{\infty} e^{z y} dF(y|\Phi_{t-1}) = \sum_{i=1}^{k} \int_{-\infty}^{\infty} e^{z y} dF(y|\Phi_{t-1}) \tag{2.6}
\]

As in Bühlmann et al. (1996), we define a sequence \( \{\Lambda_t\}_{t \in T} \) with \( \Lambda_0 = 1 \) and

\[
\Lambda_t = \prod_{k=1}^{t} \frac{e^{\theta_k z}}{M_{Y_k|\Phi_k|\Phi_{k-1}}(\Theta_k)}, \quad t \in T \setminus \{0\}. \tag{2.7}
\]

Then, \( \{\Lambda_t\}_{t \in T} \) is a martingale under \( P \) with respect to \( \Phi \). Note that the martingale property for \( \{\Lambda_t\}_{t \in T} \) still holds under the general \( k \)-regime SETAR(d; \( p_1, p_2, \ldots, p_k \)) model. Let \( P_t \) denote the restriction \( P|\Phi_t \) of \( P \) on \( \Phi_t \), for each \( t \in T \setminus \{0\} \), where \( P_0 = P \). We define a family of probability measures \( \{P_{t,\Lambda_t}\}_{t \in T \setminus \{0\}} \) by the following SETARET:

\[
P_{t,\Lambda_t}(\{Y_t \in B\}|\Phi_{t-1}) = \mathbb{E}_{P_t} \left( I\{Y_t \in B\} \frac{e^{\theta_t Y_t}}{P_t(e^{\theta_t Y_t}|\Phi_{t-1})} \bigg{|} \Phi_{t-1} \right), \tag{2.8}
\]

where \( B \) is an open interval on the real line and \( I\{Y_t \in B\} \) represents the indicator function of the event \( \{Y_t \in B\} \).

Let \( F(y; \Theta_t|\Phi_{t-1}) \) be the probability distribution of \( Y_t \) given \( \Phi_{t-1} \) under \( P_{t,\Lambda_t} \). Then, \( F(y; \Theta_t|\Phi_{t-1}) \) is given by:

\[
F(y; \Theta_t|\Phi_{t-1}) = \sum_{i=1}^{k} \int_{-\infty}^{\infty} e^{\theta^{(i)}_t x} dF(x|\Phi_{t-1}) = \sum_{i=1}^{k} \int_{-\infty}^{\infty} e^{\theta^{(i)}_t x} dF(x|\Phi_{t-1}) = \sum_{i=1}^{k} \int_{-\infty}^{\infty} e^{\theta^{(i)}_t x} dF(x|\Phi_{t-1}) \tag{2.9}
\]

Let \( M_{Y_t|\Phi_{t-1}}(z; \Theta_t) \) denote the moment generating function of \( F(y; \Theta_t|\Phi_{t-1}) \). Then, \( M_{Y_t|\Phi_{t-1}}(z; \Theta_t) \) can be related to \( M_{Y_t|\Phi_{t-1}}(z) \) as follows:

\[
M_{Y_t|\Phi_{t-1}}(z; \Theta_t) = \frac{M_{Y_t|\Phi_{t-1}}(z + \Theta_t)}{M_{Y_t|\Phi_{t-1}}(\Theta_t)}. \tag{2.10}
\]

Hence,

\[
M_{Y_t|\Phi_{t-1}}(z; \Theta_t) = \sum_{i=1}^{k} \int_{-\infty}^{\infty} e^{(z + \theta^{(i)}_t) y} dF(y|\Phi_{t-1}) = \sum_{i=1}^{k} \int_{-\infty}^{\infty} e^{(z + \theta^{(i)}_t) y} dF(y|\Phi_{t-1}) = \sum_{i=1}^{k} \int_{-\infty}^{\infty} e^{(z + \theta^{(i)}_t) y} dF(y|\Phi_{t-1}) \tag{2.11}
\]

Now, we determine an equivalent martingale measure \( Q \) by adopting the SETARET. First, given \( \Phi_{t-1} \), we know \( Y_{t-1} \in (r_{i-1}, r_i] \), for some \( i \in \{1, 2, \ldots, k\} \). In this case, the model is in the \( i^{th} \) regime at time \( t \). Hence, \( \Theta_t := \theta^{(i)}_t \). To determine \( Q \), we need to choose a sequence of risk-neutralized SETAREPs \( \{\hat{\Theta}_t\}_{t \in T \setminus \{0\}} \). For each \( t \in T \setminus \{0\} \), the determination of \( \hat{\Theta}_t \) is equivalent to the determination of \( \theta^{(i)}_t \) when
the model is in the $i_t$th regime at time $t$. The parameter $\hat{\theta}_i(t)$ can be determined by solving the following equation:

$$r = \ln \left( \frac{\int_{-\infty}^{\infty} e^{(1+\hat{\theta}_i(t))y}dF(y|\sigma_{i_t}-\frac{1}{2}\sigma_{i_t}^2+\beta(i)Y_{t-1_i},\sigma_{i_t}^2)}{\int_{-\infty}^{\infty} e^{\theta_i(t)y}dF(y|\sigma_{i_t}-\frac{1}{2}\sigma_{i_t}^2+\beta(i)Y_{t-1_i},\sigma_{i_t}^2)} \right).$$

(2.12)

We can define a family of probability measures $\{P_{\hat{\Theta}_i(t)}\}_{t\in T\setminus\{0\}}$ associated with $\{\hat{\Theta}_i(t)\}_{t\in T\setminus\{0\}}$ by (2.7) and (2.8) with $\{\Theta_i(t)\}$ replaced by $\{\hat{\Theta}_i(t)\}$. Then, we set $Q := P_{\hat{\Theta}_i(t)}$. Under $Q$, the discounted stock-price process $\{e^{-rt}S_t\}_{t\in T}$ is a martingale with respect to $\Phi$. The Fundamental theorem of asset pricing was proposed by Harrison and Kreps (1979) and further developed by Harrison and Pliska (1981, 1983), Dybyig and Ross (1987), Back and Pliska (1991), Schachermayer (1992) and Delbaen and Schachermayer (1994). It states that the absence of arbitrage is equivalent to the existence of a martingale pricing measure under which the discounted asset price process is a martingale. Back and Pliska (1991) showed that the absence of arbitrage is equivalent to the existence of an equivalent martingale measure in a discrete-time and infinite-state-space setting. Delbaen and Schachermayer (1994) pointed out that the equivalence is not always true in the continuous-time setting. Hence, they adopted the term “essentially equivalent” instead of “equivalent” to describe the relationship. By using $Q$, the price for the derivative $V$ at time $t \in T$ is given by:

$$V_t = E_Q(e^{-r(T-t)}V_T|\Phi_t).$$

(2.13)

Now, we justify the option price by considering a self-exciting threshold dynamic utility maximization problem. First, for the $i$th-regime at time $t$, let $\gamma_i(t)$ denote the risk-averse parameter of a self-exciting threshold dynamic power utility function $u_t$ at time $t$, for each $t \in T\setminus\{0\}$. Here, the risk-averse parameter $\gamma_i(t)$ is $\Phi_{t-1}$-measurable. Then, the self-exciting threshold dynamic power utility function $u_t$ at time $t$ is defined as follows:

$$u_t(x) = \sum_{i=1}^{k} \left( \frac{x^{1-\gamma_i(t)}}{1-\gamma_i(t)} I\{\gamma_i(t) \neq 1\} + \ln x I\{\gamma_i(t) = 1\} \right) I\{r_{i-1} < Y_{i-1} \leq r_i\}.$$  

(2.14)

Given $\Phi_{t-1}$, we know $Y_{i-1} \in (r_{i-1}, r_i]$, for some $i_t \in \{1, 2, \ldots, k\}$. In this case, the model is in the $i_t$th regime at time $t$. Hence, the self-exciting threshold dynamic power utility function $u_t$ at time $t$ can be reduced to the following form:

$$u_t(x) = \frac{x^{1-\gamma_i(t)}}{1-\gamma_i(t)} I\{\gamma_i(t) \neq 1\} + \ln x I\{\gamma_i(t) = 1\}.$$  

(2.15)

Hence, the self-exciting threshold dynamic utility maximization problem can be reduced to the discrete-time dynamic utility maximization problem as in Siu, Tong and Yang (2004). Following Gerber and Shiu (1994, 1996), we assume that $V_t$ is the agent’s price of the derivative $V$ at time $t$ with $\hat{V}_T = V_T$, such that it is optimal for the agent not to buy or sell any unit of the derivative $V$ at time $t$. By adopting the approach used in Gerber and Shiu (1994, 1996), we justify the pricing result in the following proposition.

**Proposition 2.1.** For each $t \in T\setminus\{T\}$, $\hat{V}_t = V_t$.

**Proof.** For each fixed regime, the proof is similar to that in Gerber and Shiu (1994, 1996). For each time $t$, we only need to replace the expectations in the proof of
Gerber and Shiu (1994, 1996) by conditional expectations given \( \Phi_1 \), and exactly the same argument can be applied. Therefore, we omit the proof here.

The method based on the SETARET can be applied to determine an equivalent martingale measure for option valuation when the underlying risky asset follows a general \( k \)-regime \( SETAR(d; p_1, p_2, \ldots, p_k) \). The extension is straightforward. Hence, we do not repeat the argument.

It is worth mentioning that the SETAR model includes the class of linear autoregressive (AR) models as a special case. In particular, the first-order linear AR model is just the first-order SETAR model with only one regime. The continuous-time analog of the discrete-time first-order linear AR model is the Ornstein-Uhlenbeck (OU) process driven by a standard Brownian motion, which has been adopted in modelling the dynamics of spot interest rates. The OU process can incorporate the mean-reverting or “pull-back” behavior of the spot interest rates to a long-term interest rate level. It can also be used to model the dynamics of foreign exchange (FX) rates. Much of the literature focus on the continuous-time OU-type processes for the valuation of bonds and bond options. However, there is relatively little work on option valuation when the asset price dynamics of the underlying risky asset are governed by the discrete-time linear AR model. An advantage of using the discrete-time linear AR model rather than its continuous-time analog is that the former is easier to estimate by using some statistical/econometric techniques than the latter. Since it is not possible to hedge a contingent claim perfectly when the price dynamic of the underlying risky asset is described by a discrete-time linear AR model. The market is incomplete in general. Hence, there are infinitely many equivalent martingale measures. The SETARET developed in this section can provide a pertinent solution to choose an equivalent martingale pricing measure for option valuation in the incomplete market described by the linear AR case as well. For the case in which the innovations of the discrete-time linear AR model have a normal distribution, we are able to obtain an analytical expression for the option price. We will also address in Section 4 the rarely discussed option valuation problem under a linear AR model with a non-normal (e.g. a shifted gamma) innovation.


Tong (1990) mentioned that in principle, it is possible to combine the first generation models, such as the SETAR model and the ARCH model, to develop second generation models, which can perhaps combine certain desirable properties of the first generation models. In this section, we first consider the class of SETAR-ARCH model proposed in Tong (1990), which can combine the advantages of the SETAR models and the ARCH models. The SETAR models focus on modelling the non-linear behavior of the conditional mean of the logarithm return process \( Y \) and provide a piecewise linear approximation to the non-linear behavior. The ARCH models concentrate on modelling the conditional heteroscedasticity of the conditional variance of \( Y \), which was introduced by the seminal work of Engle (1982). Tong (1990) also proposed another second generation model, namely, a BL-ARCH model, which assumes that the innovations process of a bilinear model (BL) follows an ARCH model. Then, we will consider the double-threshold autoregressive heteroskedastic time series model (DTARCH) proposed by Li and Li (1996).

3.1. The SETAR-ARCH models. We suppose that under \( \mathcal{P} \), the dynamics of the logarithmic returns \( \{Y_t\}_{t \in \mathcal{T}} \) from the risky asset \( S \) are governed by a general
The pricing result can also be justified by the maximization of the stochastic power utility in Section 2.

The option pricing framework for the SETAR-ARCH model includes that of the AR-ARCH model as a special case. If there is only one regime in the SETAR part of the model, the coefficient of the term \( Y_{t-1} \) vanishes, the SETAR-ARCH model reduces to the ARCH model. The pricing framework for the SETAR-ARCH model also includes that for the AR-ARCH model as a special case. If there is only one regime in the SETAR part of the model, the SETAR-ARCH model reduces to the AR-ARCH model. We will discuss these special cases in Section 4.
3.2. The heteroscedastic DTARCH model. Here, we assume that under \( \mathcal{P} \), the dynamics of the logarithmic returns \( \{ Y_t \}_{t \in \mathcal{T}} \) from the risky asset \( S \) are governed by a general \( k \)-regime DTARCH (\( d; p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_k \)) model as follows:

\[
Y_t = \sum_{i=1}^{k} \left( r + \lambda \sqrt{V_t} - \frac{1}{2} V_t \right) + \frac{1}{2} \left( \sum_{j=1}^{k} \phi_j^{(i)} Y_{t-j} + \sqrt{V_t} \xi_t \right) I_{\{ r_{i-1} < Y_{t-1} \leq r_i \}},
\]

\[
V_t = \sum_{i=1}^{k} \left( \phi_0^{(i)} + \sum_{j=1}^{q_i} \phi_j^{(i)} Y_{t-j} \right) I_{\{ r_{i-1} < Y_{t-1} \leq r_i \}}, \quad t \in \mathcal{T} \setminus \{ 0 \},
\]  

(3.7)

where the conditional variance \( V_t \) of the process \( Y \) at time \( t \) is governed by a threshold ARCH (TARCH) model. Hence, the corresponding moment generating function \( M_{Y_t|\Phi_t=1}(z; \Theta_t) \) is given by:

\[
M_{Y_t|\Phi_t=1}(z; \Theta_t) = \sum_{i=1}^{k} \int_{-\infty}^{\infty} e^{i \beta^{(i)} t} dF(x | r + \lambda \sqrt{V_t} y) I_{\{ r_{i-1} < Y_{t-1} \leq r_i \}}.
\]

(3.9)

Hence, the probability distribution \( F(y; \Theta_t|\Phi_t=1) \) of \( Y_t \) given \( \Phi_{t-1} \) under \( \mathcal{P}_{t,A_t} \) is:

\[
F(y; \Theta_t|\Phi_t=1) = \sum_{i=1}^{k} \int_{-\infty}^{\infty} e^{i \beta^{(i)} t} dF(x | r + \lambda \sqrt{V_t} y) I_{\{ r_{i-1} < Y_{t-1} \leq r_i \}}.
\]

(3.10)

Suppose \( Y_{t-1} \in (r_{i-1}, r_{i}] \), for some \( i \in \{1, \ldots, k\} \). Then, the parameter \( \beta^{(i)} t \) can be determined by:

\[
\beta^{(i)} t = \ln \left( \int_{-\infty}^{\infty} e^{i \beta^{(i)} t} dF(y | r + \lambda \sqrt{V_t} y) I_{\{ r_{i-1} < Y_{t-1} \leq r_i \}} \right)^{\frac{1}{2}}.
\]

(3.11)

We can determine and justify the price by following the procedure in the previous sections.

The pricing method can also be applied to price options when the price dynamic of the underlying risky asset is governed by a threshold ARCH (TARCH) model. However, the option pricing problem under the class of TARCH models has been considered in the article by Härdle and Hafner (2000). Hence, we are not going to pursue this problem here.

4. Some Parametric Cases. We deal with two interesting parametric cases for the conditional distribution of the innovations \( \xi_t \) given \( \Phi_{t-1} \) of the first and the second generation models. First, we consider the case of the conditional normal distribution for the innovations. Then, we concentrate on the conditional shifted gamma distribution for the innovations. We will also provide discussion on some
Option Pricing under Threshold Autoregressive Models

4.1. Conditional normality. First, we consider the first generation model with the innovations having a normal distribution; that is, \( \{ \xi_t \}_{t \in \mathcal{T}} \) is a sequence of i.i.d. random variables, where \( \xi_t \sim N(0, 1) \), for each \( t \in \mathcal{T} \) under \( \mathcal{P} \). Given \( \Phi_{t-1} \), we suppose that the TAR model for the return is in the \( i_{th} \)-regime. Hence, the conditional distribution of \( Y_t \) given \( \Phi_{t-1} \) is a normal distribution with mean \( r + \lambda \sigma_{i_t} - \frac{1}{2} \beta^{(i_t)} Y_{t-1} \) and variance \( \sigma^2_{i_t} \). Hence, the risk-neutralized SETAREP \( \tilde{\theta}^{(i_t)}_t \) at time \( t \) is given by:

\[
\tilde{\theta}^{(i_t)}_t = -\frac{\lambda \sigma_{i_t} - \beta^{(i_t)} Y_{t-1}}{\sigma^2_{i_t}}.
\]

The moment generating function of the conditional distribution of \( Y_t \) given \( \Phi_{t-1} \) under \( \mathcal{Q} \) can be obtained as follows:

\[
M_{Y_t|\Phi_{t-1}}(z; \tilde{\theta}^{(i_t)}_t) = \exp \left[ \left( r - \frac{1}{2} \sigma^2_{i_t} \right) z + \frac{1}{2} z^2 \sigma^2_{i_t} \right].
\]

Hence, the conditional distribution of \( Y_t \) given \( \Phi_{t-1} \) under \( \mathcal{Q} \) is a normal distribution with mean \( r - \frac{1}{2} \sigma^2_{i_t} \) and variance \( \sigma^2_{i_t} \). The risk-neutralized conditional distribution of \( Y_t \) given \( \Phi_{t-1} \) will be used for simulating the terminal stock price for the computation of the option price.

Now, we consider the SETAR-ARCH model with normally distributed innovations. In this case, the conditional distribution of \( Y_t \) given \( \Phi_{t-1} \) is a normal distribution with mean \( r + \lambda \sqrt{V_t} - \frac{1}{2} V_t + \beta^{(i_t)} Y_{t-1} \) and variance \( V_t \). Hence, the risk-neutralized SETAREP \( \tilde{\theta}^{(i_t)}_t \) at time \( t \) is given by:

\[
\tilde{\theta}^{(i_t)}_t = -\lambda \sqrt{V_t} - \beta^{(i_t)} Y_{t-1}.
\]

The moment generating function of the conditional distribution of \( Y_t \) given \( \Phi_{t-1} \) under \( \mathcal{Q} \) can be obtained as follows:

\[
M_{Y_t|\Phi_{t-1}}(z; \tilde{\theta}^{(i_t)}_t) = \exp \left[ \left( r - \frac{1}{2} V_t \right) z + \frac{1}{2} z^2 V_t \right].
\]

Therefore, the conditional distribution of \( Y_t \) given \( \Phi_{t-1} \) under \( \mathcal{Q} \) is a normal distribution with mean \( r - \frac{1}{2} V_t \) and variance \( V_t \), which do not depend on the regime at time \( t \). The parametric form of the conditional variance process \( \{ V_t \}_{t \in \mathcal{T}} \) is unaltered by changing the probability measures from \( \mathcal{P} \) to \( \mathcal{Q} \).

Similarly, in the case of the DTARCH model with normally distributed innovations, the conditional distribution of \( Y_t \) given \( \Phi_{t-1} \) under \( \mathcal{Q} \) is a normal distribution with mean \( r - \frac{1}{2} V_{t-1}^{i_t} \) and variance \( V_{t-1}^{i_t} \). The dynamics of the conditional variances under \( \mathcal{Q} \) are given by:

\[
V_t^{i_t} := \phi_0^{(i_t)} + \sum_{j=1}^{q_{i_t}} \phi_j^{(i_t)} Y_{t-j}^2.
\]

However, changing probability measures from \( \mathcal{P} \) to \( \mathcal{Q} \) turns a second-generation model, namely, the SETAR-ARCH model, to a first-generation model, namely, a non-zero drift ARCH model.
4.2. **Conditional shifted Gamma distribution.** We consider the first generation models and the second generation models with innovations having a shifted gamma distribution, which can incorporate the skewed behavior in the conditional distribution of the stock innovations. First, we define a sequence of i.i.d. random variables \( \{X_t\}_{t \in \mathcal{T} \setminus \{0\}} \), where the common distribution for \( X_t \) is a gamma distribution \( Ga(a, b) \) with shape parameter \( a \) and scale parameter \( b \); that is, the density function \( f_{X_t}(x) \) of \( X_t \) is given by:

\[
f_{X_t}(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}, \quad t \in \mathcal{T} \setminus \{0\}.
\]

(4.6)

Then, we suppose that the innovations process \( \{\xi_t\}_{t \in \mathcal{T}} \) is given as follows:

\[
\xi_t = \frac{X_t - \frac{r}{\sqrt{\beta^2}}}{\sqrt{\sigma^2}}.
\]

(4.7)

First, we deal with the first generation model with shifted gamma distributed innovations. In this case, we can write the dynamics of the logarithmic returns \( \{Y_t\}_{t \in \mathcal{T}} \) as follows:

\[
Y_t = \sum_{i=1}^{k} \left( r + \lambda \sigma_i - \frac{1}{2} \sigma_i^2 + \beta^{(i)} Y_{t-1} - \sqrt{a \sigma_i^2} + b \sqrt{\frac{\sigma_i^2}{a}} X_t \right) I_{\{r_i \leq Y_{t-1} \leq r_{i+1}\}}.
\]

(4.8)

Given \( \Phi_{t-1} \), the TAR model for the return is in the \( i_t \)th-regime. Then, the conditional distribution of \( b \sigma_i \sqrt{\frac{1}{a}} X_t \) given \( \Phi_{t-1} \) is a gamma distribution \( Ga(a, \sqrt{\frac{\mu_i}{\sigma_i^2}}) \) with shape parameter \( a \) and scale parameter \( \frac{\mu_i}{\sigma_i^2} \). The conditional distribution of \( Y_t \) given \( \Phi_{t-1} \) is a shifted gamma distribution with shape parameter \( a \), scale parameter \( \frac{\sqrt{\mu_i}}{\sigma_i} \) and shifted parameter \( -r - \lambda \sigma_i + \frac{1}{2} \sigma_i^2 - \beta^{(i)} Y_{t-1} + \sqrt{a \sigma_i^2} \). Hence, the moment generating function of \( Y_t \) given \( \Phi_{t-1} \) under \( \mathcal{P} \) is:

\[
M_{Y_t|\Phi_{t-1}}(\theta) = \left( \frac{\sqrt{\frac{\mu_i}{\sigma_i^2}}}{\sqrt{\frac{\mu_i}{\sigma_i^2}} - \theta} \right)^a \exp\left( (r + \lambda \sigma_i - \frac{1}{2} \sigma_i^2 + \beta^{(i)} Y_{t-1} - \sqrt{a \sigma_i^2}) \theta \right), \quad \theta < \sqrt{\frac{\mu_i}{\sigma_i^2}}.
\]

(4.9)

Then, the moment generating function of \( Y_t \) given \( \Phi_{t-1} \) under \( \mathcal{Q} \) is:

\[
M_{Y_t|\Phi_{t-1}}(z; \theta_t) = \left( \sqrt{\frac{\mu_i}{\sigma_i^2} - \theta_t} \right)^a \exp\left( (r + \lambda \sigma_i - \frac{1}{2} \sigma_i^2 + \beta^{(i)} Y_{t-1} - \sqrt{a \sigma_i^2}) z \right),
\]

provided that \( z < \sqrt{\frac{\mu_i}{\sigma_i^2} - \theta_t} \).

Hence, the risk-neutralized SETAREP at time \( t \) is given by:

\[
\hat{\theta}^{(i)}_t = \sqrt{\frac{\mu_i}{\sigma_i^2}} - \left[ 1 - \exp\left( \frac{\lambda \sigma_i - \frac{1}{2} \sigma_i^2 + \beta^{(i)} Y_{t-1} - \sqrt{a \sigma_i^2}}{a} \right) \right]^{-1}.
\]

(4.11)
Write \( \hat{b}_t^{(i)} \) for the scale parameter \( \sqrt{\frac{a}{\sigma_t^2}} \) of the shifted gamma distribution under \( \mathcal{P} \). Then, the risk-neutralized scale parameter \( \hat{b}_t^{(i)} \) is given by:

\[
\hat{b}_t^{(i)} := b_t^{(i)} - \hat{b}_t^{(i)} = \left[ 1 - \exp \left( \frac{\lambda \sigma_t - \frac{1}{2} \sigma_t^2 + \beta^{(i)} Y_{t-1}}{a} \right) \right]^{-1}, \quad t \in T \setminus \{0\}
\]  

Therefore, under \( \mathcal{Q} \), \( Y_t|\Phi_{t-1} \) follows a shifted gamma distribution with shape parameter \( a \), scale parameter \( \hat{b}_t^{(i)} \) and shifted parameter \( -r - \lambda \sigma_t + \frac{1}{2} \sigma_t^2 - \beta^{(i)} Y_{t-1} + \sqrt{a \sigma_t^2} \). The change of probability measures only alters the real-world scale parameter \( \hat{b}_t^{(i)} \) to the risk-neutralized one \( \hat{b}_t^{(i)} \).

For the SETAR-ARCH model with shifted gamma distributed innovations, we follow the above procedure, except replacing \( \sigma_x^2 \) with \( V_t \), and the conditional distribution of \( Y_t \) given \( \Phi_{t-1} \) under \( \mathcal{Q} \) is a shifted gamma distribution with shape parameter \( a \), scale parameter \( \hat{b}_t^{(i)} \) and shifted parameter \( -r - \lambda \sqrt{V_t} + \frac{1}{2} V_t - \beta^{(i)} Y_{t-1} + \sqrt{a V_t} \), where \( \hat{b}_t^{(i)} \) is given by:

\[
\hat{b}_t^{(i)} = \left[ 1 - \exp \left( \frac{\lambda \sqrt{V_t} - \frac{1}{2} V_t + \beta^{(i)} Y_{t-1} - \sqrt{a V_t}}{a} \right) \right]^{-1}, \quad t \in T \setminus \{0\}.
\]  

The parametric form of the process \( \{V_t\}_{t \in T} \) is unaltered by the change of measures. However, it is no longer the conditional variance process under \( \mathcal{Q} \). In fact, the conditional variance process \( \{\hat{V}_t^{(i)}\}_{t \in T} \) under \( \mathcal{Q} \) is given by:

\[
\hat{V}_t^{(i)} = \frac{a^2}{\hat{b}_t^{(i)}}, \quad t \in T \setminus \{0\},
\]  

Unlike the case of the SETAR-ARCH model with normally distributed innovations, the conditional distribution of \( Y_t \) given \( \Phi_{t-1} \) under \( \mathcal{Q} \) depends on the regime \( i_t \) of the SETAR model at time \( t \).

Similarly, for the DURATION model with shifted gamma innovations, the conditional distribution of \( Y_t \) given \( \Phi_{t-1} \) under \( \mathcal{Q} \) is a shifted gamma distribution with shape parameter \( a \), scale parameter \( \hat{b}_t^{(i)} \) and shifted parameter \( -r - \lambda \sqrt{V_t} + \frac{1}{2} V_t - \beta^{(i)} Y_{t-1} + \sqrt{a V_t} \), where \( \hat{b}_t^{(i)} \) is given by:

\[
\hat{b}_t^{(i)} = \left[ 1 - \exp \left( \frac{\lambda \sqrt{V_t} - \frac{1}{2} V_t + \beta^{(i)} Y_{t-1} - \sqrt{a V_t}}{a} \right) \right]^{-1}, \quad t \in T \setminus \{0\}.
\]  

Again, the parametric forms of the processes \( \{V_t\}_{t \in T} \) and \( \{\hat{V}_t^{(i)}\}_{t \in T} \) are unaltered by the change of measures from \( \mathcal{P} \) to \( \mathcal{Q} \). The conditional variance process \( \{\hat{V}_t^{(i)}\}_{t \in T} \) under \( \mathcal{Q} \) is given by:

\[
\hat{V}_t^{(i)} = \frac{a^2}{\hat{b}_t^{(i)}}, \quad t \in T \setminus \{0\}.
\]
4.3. Other special cases. First, we consider the case that the dynamics of the logarithmic returns \( \{ Y_t \}_{t \in \mathcal{T}} \) are governed by the following discrete-time linear AR(p) model:

\[
Y_t = r + \lambda \sigma - \frac{1}{2} \sigma^2 + \sum_{k=1}^{p} \beta_k Y_{t-k} + \sigma \xi_t ,
\]

where \( \{ \xi_t \}_{t \in \mathcal{T}} \) are i.i.d. standard normal random variables.

This is a special case of the SETAR model with only one regime. In this case, the risk-neutral distribution of \( Y_t \) given \( \Phi_{t-1} \) is a normal distribution with mean \( r - \frac{1}{2} \sigma^2 \) and variance \( \sigma^2 \). Hence, a closed-form pricing formula for a standard European call option with strike price \( K \) and maturity at time \( T \) is given by the following discrete-time version of the celebrated Black-Scholes formula:

\[
C(S_t) = S_t N(d_1) - K e^{-r(T-t)} N(d_2) ,
\]

where

\[
d_1 = \frac{\ln(S_t/K) + r + \frac{1}{2} \sigma^2(T-t)}{\sigma \sqrt{T-t}} ,
\]

\[
d_2 = d_1 - \sigma \sqrt{T-t} ,
\]

and \( N(\cdot) \) is the probability distribution function of a standard normal distribution.

Now, we deal with the discrete-time linear AR(1) model with shifted gamma distributed innovations. The same procedure can be applied to deal with the case of a linear AR(p) model. We suppose that the dynamics of the logarithmic returns \( \{ Y_t \}_{t \in \mathcal{T}} \) are governed by:

\[
Y_t = r + \lambda \sigma - \frac{1}{2} \sigma^2 + \beta Y_{t-1} - \sqrt{a} \sigma + \sqrt{a} X_t .
\]

By using the result in Section 4.2, it can be shown that under \( Q \), \( Y_t | \Phi_{t-1} \) follows a shifted gamma distribution with shape parameter \( a \), scale parameter \( \hat{b}_t \) and shifted parameter \( -r - \lambda \sigma + \frac{1}{2} \sigma^2 + \beta Y_{t-1} - \sqrt{a} \sigma \), where the risk-neutral scale parameter \( \hat{b}_t \) is given by:

\[
\hat{b}_t = \left[ 1 - \exp \left( \frac{\lambda \sigma - \frac{1}{2} \sigma^2 + \beta Y_{t-1} - \sqrt{a} \sigma}{a} \right) \right]^{-1} , \quad t \in T \setminus \{0\} .
\]

In either the ARCH model with normal innovations or the AR-ARCH model with normal innovations, the risk-neutral conditional distribution for \( Y_t \) given \( \Phi_{t-1} \) and the corresponding conditional volatility under \( Q \) are the same as those for the SETAR-ARCH model.

For the ARCH(q) model with shifted gamma distributed innovations, the risk-neutral distribution for \( Y_t \) given \( \Phi_{t-1} \) is again a shifted gamma distribution with shape parameter \( a \), scale parameter \( \hat{b}_t \) and shifted parameter \( -r - \lambda \sqrt{V_t} + \frac{1}{2} V_t + \sqrt{a} V_t \), where \( \hat{b}_t \) is given by:

\[
\hat{b}_t = \left[ 1 - \exp \left( \frac{\lambda \sqrt{V_t} - \frac{1}{2} V_t - \sqrt{a} V_t}{a} \right) \right]^{-1} , \quad t \in T \setminus \{0\} .
\]

Again, the parametric form of the process \( \{ V_t \}_{t \in \mathcal{T}} \) is unaltered by the change of measures and is given by:

\[
V_t = \phi_0 + \sum_{j=1}^{q_j} \phi_j Y_{t-j}^2 .
\]
The conditional variance process \( \{ \hat{V}_t \}_{t \in T} \) under \( Q \) is given by:

\[
\hat{V}_t = a^2 \left[ 1 - \exp \left( \frac{\lambda \sqrt{V_t} - \frac{1}{2} V_t - \sqrt{aV_t}}{a} \right) \right].
\] (4.23)

Finally, we suppose that the dynamics of \( \{ Y_t \}_{t \in T} \) are governed by the following AR(1)-ARCH(q) model:

\[
Y_t = r + \lambda \sqrt{V_t} - \frac{1}{2} V_t + \beta Y_{t-1} - \sqrt{aV_t} + b \sqrt{V_t} X_t,
\]

\[
V_t = \phi_0 + \sum_{j=1}^{q} \phi_j Y_{t-j}^2, \quad t \in T \setminus \{0\}.
\] (4.24)

Then, the risk-neutral distribution for \( Y_t \) given \( \Phi_{t-1} \) is again a shifted gamma distribution with shape parameter \( a \), scale parameter \( \hat{b}_t \) and shifted parameter \( -r - \lambda \sqrt{V_t} + \frac{1}{2} V_t + \beta Y_{t-1} + \sqrt{aV_t} \), where \( \hat{b}_t \) is given by:

\[
\hat{b}_t = \left[ 1 - \exp \left( \frac{\lambda \sqrt{V_t} - \frac{1}{2} V_t + \beta Y_{t-1} - \sqrt{aV_t}}{a} \right) \right]^{-1}, \quad t \in T \setminus \{0\}.
\] (4.25)

The conditional variance process \( \{ \hat{V}_t \}_{t \in T} \) under \( Q \) is given by:

\[
\hat{V}_t = a^2 \left[ 1 - \exp \left( \frac{\lambda \sqrt{V_t} - \frac{1}{2} V_t + \beta Y_{t-1} - \sqrt{aV_t}}{a} \right) \right].
\] (4.26)

5. Simulation Studies for Option Prices. In this section, we investigate consequences for the option prices of the threshold effect described by the first generation models (i.e. the TAR models) and the threshold effect described by the second generation models (i.e. the DTARCH models) under two parametric assumptions, namely, the conditional normal innovations and the conditional shifted gamma innovations. In particular, we focus on the qualitative behaviors of the option prices versus various strike prices when the threshold effect is described by the first-generation model and the second-generation model in which the conditional heteroskedastic effect is present.

We simulate the European call option prices from the TAR models and the DTARCH models under different configurations of the hypothetical specimen parameter values and different parametric assumptions for the stock innovations, such as the normal innovations and the shifted gamma innovations. We employ the Monte Carlo simulation coupled with the control variate technique for computing the option prices. 10,000 simulation runs will be generated for computing each option price implied by each first-generation model based on a particular configuration of the model parameters. More simulation runs are required to obtain accurate approximation results for the option prices implied by the second-generation model. In particular, we will generate 20,000 and 50,000 simulation runs for computing each option price implied by each second-generation model with normal innovations and shifted-gamma innovations, respectively. We only consider the first-order two-regime TAR model and the first-order two-regime DTARCH model with the delay parameter \( d = 1 \) in both cases. We consider the specimen values of the model parameters and suppose that the annual risk-free interest rate is 10% (i.e. the daily interest rate is 0.000378 for 252 trading days per annum); the daily unit-risk premium is 0.000378; the option maturity is four months, which is equivalent to 84 trading days. We did the computations by C++ codes with GSL functions. The
simulation results will be discussed and presented in the sequel. All figures for the simulation results for the option prices are displayed in the Appendix.

5.1. The impact of the threshold effect in the first-generation model. We investigate the impact of the threshold effect described by the first-order two-regime TAR model on the option prices. First, we consider the case of the conditional normal innovations and the specimen values for the model parameters. We assume that the threshold level \( r_1 = 0.000378 \), which is equal to the risk-free interest rate; that is, if the logarithmic return in the current period is greater than the risk-free interest rate, the logarithmic return in the next period follows the second-regime dynamics; otherwise, it follows the first-regime dynamics. We make this choice of the threshold level \( r_1 \) for the purpose of illustration. We further suppose that the volatility rates \( \sigma_1 \) and \( \sigma_2 \) in the first and the second regimes are 0.0156 and 0.0312, respectively. The BS volatility is given by the average of \( \sigma_1 \) and \( \sigma_2 \). We compare the option prices implied by the TAR model with conditionally normal innovations and those implied by the AR model with conditionally normal innovations. Note that the option prices obtained from the AR model are the same as the Black-Scholes prices. Figure 1 presents the plots of the call prices implied by the TAR model against the strike prices (the wiggly curve) and the Black-Scholes call prices against the strike prices (the solid curve). As the strike price increases, both the TAR call prices and the BS call prices decrease. The TAR call prices are always greater than the corresponding Black-Scholes prices for various strike prices. Since the threshold level is the risk-free interest rate, it is more likely that the logarithmic return of the risky asset at a particular period is greater than the threshold level. Hence, it is more likely that the logarithmic return of the risky asset in the next period follows the second-regime dynamics, which has a higher level of volatility than the average of the volatilities in the two regimes, which is the Black-Scholes volatility here. This can explain why the option prices implied by the TAR model are greater than the corresponding one implied by the Black-Scholes model. The choice of the threshold level as the risk-free interest rate can make the threshold effect on the option prices implied by the TAR model more apparent.

Now, we consider the case of conditional shifted-gamma innovations. In this case, the option prices implied by the AR model are different from those implied from the BS model. We consider the specimen values for the model parameters and assume that \( r_1 = 0.000378 \); the shape parameter \( \alpha = 100 \); the daily unit risk premium \( \lambda = 0.000378 \); the autoregressive parameters in the conditional mean \( \beta^{(1)} = \beta = 0.1 \), \( \beta^{(2)} = 0.3 \); \( \sigma_1 = \sigma = 0.0156 \) and \( \sigma_2 = 0.0312 \). Figure 2 presents the plots of the call prices implied by the TAR model with shifted-gamma innovations against the strike prices (the lighted-wiggly curve), the linear AR model with shifted-gamma innovations against the strike prices (the thicker-wiggly curve) and the Black-Scholes call prices against the strike prices (the solid curve). As the strike price increases, the call prices implied by all of the three models decrease. The call prices implied by the TAR model are always greater than the corresponding Black-Scholes prices for various strike prices while the call prices implied by the linear AR model are always less than the corresponding Black-Scholes prices for various strike prices. These results are consistent with those in the case of conditional normal innovations.

Both the results of the conditional normal innovations and the conditional shifted gamma innovations reveal that the introduction of the threshold effect in the conditional mean by the first-generation model increases the call prices for various strike prices.
5.2. Consequences of the threshold effect in the second generation model.

We study consequences for the option prices of the threshold effect in the conditional variance process described by the first-order two regimes DTARCH model. We compare the option prices implied by the first-order two-regime DTARCH model with those implied from the SETAR-ARCH model, in which the threshold effect in the conditional variance process is not incorporated. First, we consider the case of conditional normal innovations and the specimen values for the model parameters.

We assume that the threshold level \( r_1 = 0.000378 \); \( \phi_0^{(1)} = \phi_0 = 0.003; \) \( \phi_1^{(1)} = \phi_1 = 0.63; \) \( \phi_0^{(2)} = 0.006; \) \( \phi_1^{(2)} = 0.36 \). Note that the option prices obtained from the SETAR-ARCH model are the same as those obtained from the AR-ARCH model.

The stationary volatilities in the first-regime and second-regime ARCH processes are given by 0.090045 and 0.096825, respectively. The average of them is 0.093435. For the control variates technique, we adopt the theoretical BS call prices and the simulated BS call prices with the constant volatility level specified by the average of the stationary volatilities for the first-regime ARCH process and the second-regime ARCH process to adjust for the simulated call prices implied by the DTARCH model. We use the theoretical BS call prices and the simulated BS call prices with the constant volatility level specified the stationary volatility for the first-regime ARCH process to adjust for the simulated call prices implied by the SETAR-ARCH model.

Figure 3 presents the plots of the call prices implied by the DTARCH model with normal innovations against strike prices (the wiggly curve) and those obtained from the SETAR-ARCH model with normal innovations against strike prices (the solid curve). As the strike price increases, both the DTARCH call prices and the SETAR-ARCH call prices decrease. The call prices implied by the DTARCH model are always less than the corresponding call prices implied by the SETAR-ARCH model without the threshold effect in the conditional variances for various strike prices.

Now, we consider the case that the innovations follow a shifted gamma distribution. In this case, the option prices implied by the AR model are different from those implied by the BS model. We consider the specimen values for the model parameters and assume that \( r_1 = 0.000378 \); the shape parameter \( a = 100 \); the daily unit risk premium \( \lambda = 0.000378 \); the autoregressive parameters in the conditional mean \( \beta^{(1)} = \beta = 0.1; \) \( \beta^{(2)} = 0.3; \) \( \phi_0^{(1)} = \phi_0 = 0.003; \) \( \phi_1^{(1)} = \phi_1 = 0.63; \) \( \phi_0^{(2)} = 0.006; \) \( \phi_1^{(2)} = 0.36 \). Figure 4 presents the plots of the call prices implied by the DTARCH model with shifted gamma innovations against strike prices (the wiggly curve) and those obtained from the SETAR-ARCH model with shifted gamma innovations against strike prices (the solid curve). As the strike price increases, both the DTARCH call prices and the SETAR-ARCH call prices decrease. As in the case of the first-generation models with shifted gamma innovations, the call
prices implied by the DTARCH model with shifted gamma innovations are greater than the corresponding call prices implied by the SETAR-ARCH model with shifted gamma innovations for most of the values for the strike prices. This reveals that the threshold effect in the conditional means described by the first-generation model has more significant impact on the option prices than the threshold effect in the conditional variances described by the second-generation model.

6. Further Research. For further investigation, it would be interesting to consider the extensions of our pricing model to deal with American options, Asian options, interest rate derivatives and credit derivatives. Applications of our model to price various kinds of reinsurance products which exhibit “derivative” feature is a topic of practical interest in the insurance industries. We may also consider the option pricing problem under different types of second generation models, third generations, and so on, like SETAR-EGARCH models. It is interesting to conduct empirical investigation on the performance of our model for different asset classes, like exchange rates and interest rates, using high frequency data.

7. Appendix. see Figure 1, 2, 3 and 4.

Acknowledgment. We would like to thank the referee for helpful comments and suggestions. This work was supported by Research Grants Council of HKSAR (Project No: HKU 7050/05P).

REFERENCES

Figure 1. Impact of the threshold effect described by the TAR model with normal innovations on the option prices.

Figure 2. Impact of the threshold effect described by the TAR model with shifted gamma innovations on the option prices.


Figure 3. Impact of the threshold effect described by the DTARCH model with normal innovations on the option prices

Figure 4. Impact of the threshold effect described by the DTARCH model with shifted-gamma innovations on the option prices

OPTION PRICING UNDER THRESHOLD AUTOREGRESSIVE MODELS


Received September 2005; Revised January 2006

E-mail address: T.K.Siu@ma.hw.ac.uk
E-mail address: h.tong@lse.ac.uk
E-mail address: hlyang@hkusua.hku.hk