DIAGRAMMATIC CHARACTERISATION OF ENRICHED ABSOLUTE COLIMITS

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ABSTRACT. We provide a diagrammatic criterion for the existence of an absolute colimit in the context of enriched category theory.

An absolute colimit is one preserved by any functor; the class of absolute colimits was characterised for ordinary categories by Paré [4] and for enriched ones by Street [5]. For categories enriched over a monoidal category $V$ or bicategory $W$, the appropriate colimits are the weighted colimits of [6], and Street’s characterisation is in fact one of the class of absolute weights: those weights $\varphi$ such that $\varphi$-weighted colimits are preserved by any functor. This is different to Paré’s result, which gives a diagrammatic characterisation of when a particular cocone is absolutely colimiting. In this note, we give a result in the enriched context which is closer in spirit to Paré’s than to Street’s. This result is very useful in practice, but seems not to be in the literature; we set it down for future use.

1. The result

1.1. Background. We work in the context of bicategory-enriched category theory; see [6], for example. $W$ will denote a bicategory whose homs are locally small, complete and cocomplete categories, and which is biclosed, meaning that for each 1-cell $A: x \to y$ in $W$, the composition functors $A \otimes (-): W(z, x) \to W(z, y)$ and $(-) \otimes A: W(y, z) \to W(x, z)$ have right adjoints $[A, -]$ and $\langle A, - \rangle$ respectively.

A $W$-category $\mathcal{A}$ comprises a set $\text{ob}\mathcal{A}$ of objects; for each $a \in \text{ob}\mathcal{A}$ an object $\epsilon a \in \text{ob}W$, the extent of $a$; for each pair of objects $a, b$, a hom-object $C(b, a) \in W(\epsilon a, \epsilon b)$; and identity and composition 2-cells $\iota: I_a \to C(a, a)$ and $\mu: C(c, b) \otimes C(b, a) \to C(c, a)$ satisfying the expected axioms. A $W$-profunctor $M: \mathcal{A} \to \mathcal{B}$ is given by objects $M(b, a) \in W(\epsilon a, \epsilon b)$ and action maps $\mu: B(b', b) \otimes M(b, a) \otimes \mathcal{A}(a, a') \to M(b', a')$ satisfying unitality and associativity axioms. A profunctor map $M \to M': \mathcal{A} \to \mathcal{B}$ comprises maps $M(b, a) \to M'(b, a)$ compatible with the actions by $\mathcal{A}$ and $\mathcal{B}$. The identity profunctor $\mathcal{A}: \mathcal{A} \to \mathcal{A}$ has components $\mathcal{A}(b, a)$ with action given by composition in $\mathcal{A}$. For profunctors $M: \mathcal{A} \to \mathcal{B}$ and $N: \mathcal{B} \to \mathcal{C}$ with $\mathcal{B}$ small, the tensor product $N \otimes_{\mathcal{B}} M: \mathcal{A} \to \mathcal{C}$
preserving assignation on objects, together with 2-cells $X$ has components given by coequalisers

and actions by $C$ and $A$ inherited from $N$ and $M$. Small $W$-categories, profunctors and profunctor maps comprise a bicategory $W$-Mod. There is a full embedding $W 	o W$-Mod sending $X$ to the $W$-category $X$ with one object $*$ with $ε(*) = X$ and $X(*, *) = I_X$.

If $A$ and $B$ are $W$-categories, then a $W$-functor $F: A \to B$ comprises an extent-preserving assignation on objects, together with 2-cells $C(b, a) \to D(Fb, Fa)$ subject to two functoriality axioms. If $F: A \to C$ and $G: B \to C$ are $W$-functors then there is an induced profunctor $C(G, F): A \to B$ with components $C(G, F)(b, a) = C(Gb, Fa)$ and action derived from the action of $F$ and $G$ on homs and composition in $C$.

Given profunctors $M: A \to B, N: B \to C$ and $L: A \to C$ with $B$ small, a profunctor map $u: N \otimes B M \to L$ is said to exhibit $M$ as $[N, L]$ if every map $f: N \otimes B K \to L$ is of the form $u \circ (N \otimes B ʃ)$ for a unique $ʃ: K \to M$; while it is said to exhibit $N$ as $⟨M, L⟩$ if every $f: K \otimes B M \to L$ is of the form $u \circ (ʃ \otimes B M)$ for a unique $ʃ: K \to N$.

Given $ϕ: A \to B$ in $W$-Mod and a functor $F: B \to C$, a $ϕ$-weighted colimit of $F$ is a functor $Z: A \to C$ and profunctor map $a: ϕ \to C(F, Z)$ such that for each $C \in C$, the map

$$ϕ \otimes A C(Z, C) \xrightarrow{a \otimes A 1} C(F, Z) \otimes A C(Z, C) \xrightarrow{μ} C(F, C)$$

(1)

exhibits $C(Z, C)$ as $[ϕ, C(F, C)]$. A functor $G: C \to D$ preserves this colimit just when the composite $ϕ \to C(F, Z) \to D(GF, GZ)$ exhibits $GZ$ as a $ϕ$-weighted colimit of $GF$; the colimit is absolute when it is preserved by all functors out of $C$. [5] proves that $ϕ$-weighted colimits are absolute if and only if $ϕ$ admits a right adjoint in $W$-Mod.

Dually, given $ψ: B \to A$ in $W$-Mod and a functor $F: B \to C$, a $ψ$-weighted limit of $F$ is a functor $Z: A \to C$ and map $b: ψ \to C(Z, F)$ such that for each $C \in C$, the map

$$C(C, Z) \otimes A ψ \xrightarrow{1 \otimes A b} C(C, Z) \otimes A C(Z, F) \xrightarrow{μ} C(C, F)$$

exhibits $C(C, Z)$ as $⟨ψ, C(C, Z)⟩$. Absoluteness of limits is defined as before; now every limit weighted by $ψ: B \to A$ is absolute if and only if $ψ$ has a left adjoint in $W$-Mod.

1.2. Theorem. Let $ϕ: A \to B$ admit the right adjoint $ψ: B \to A$ in $W$-Mod, and let $F: B \to C$ and $Z: A \to C$ be $W$-functors. There is a bijective correspondence between data of the following forms:

(a) A map $a: ϕ \to C(F, Z)$ exhibiting $Z$ as a $ϕ$-weighted colimit of $F$;
(b) A map $b: ψ \to C(Z, F)$ exhibiting $Z$ as a $ψ$-weighted limit of $F$;
(c) Maps $a: ϕ \to C(F, Z)$ and $b: ψ \to C(Z, F)$ such that the following two squares commute in $W$-Mod$(A, A)$ and $W$-Mod$(B, B)$:

$$\begin{array}{ccc}
A & \xrightarrow{η} & ψ \otimes B ϕ \\
\downarrow Z & & \downarrow b \otimes B a \\
C(Z, Z) & \xleftarrow{μ} & C(Z, F) \otimes B C(F, Z)
\end{array} \quad \begin{array}{ccc}
ϕ \otimes A ψ & \xrightarrow{ε} & B \\
\downarrow a \otimes A b & & \downarrow F \\
C(F, Z) \otimes A C(Z, F) & \xrightarrow{μ} & C(F, F)
\end{array}$$

(2)
1.3. Examples.

We first consider examples wherein responding to a monoidal category $V$ for existence, we let $\bar{\cdot}$ for surjectivity, suppose given $a$ short calculation using commutativity in the right square and the triangle identities.

Evaluating in the second variable at any $a$ follows easily that (3) exhibits $C(Z,F)$ as $[\varphi, C(F,F)]$. Applying this universality to the composite $\varepsilon \circ F$: $\varphi \otimes_A \psi \rightarrow B \rightarrow C(F,F)$ yields a unique map $b: \psi \rightarrow C(Z,F)$ making the right square of (2) commute; we must show that the left one does too. Arguing as before shows that

$$\varphi \otimes_A C(Z,Z) \xrightarrow{a \otimes \varphi 1} C(F,Z) \otimes_A C(Z,F) \xrightarrow{\mu} C(F,F).$$

(3)

exhibits $C(Z,Z)$ as $[\varphi, C(F,F)]$. It thus suffices to show that the left square of (2) commutes after applying the functor $\varphi \otimes_A (-)$ and postcomposing with (4); which follows by a short calculation using commutativity in the right square and the triangle identities.

So from the data in (a) we may obtain that in (c), and the assignation is injective, since $b$ is uniquely determined by universality of $a$ and commutativity on the right of (2). For surjectivity, suppose given $a$ and $b$ as in (c); we must show that $a$ exhibits $Z$ as a $\varphi$-weighted colimit of $F$, in other words, that for each $C \in C$, the map (1) exhibits $C(Z,C)$ as $[\varphi, C(F,C)]$, or in other words, that for each map $f: \varphi \otimes_A K \rightarrow C(F,C)$, there is a unique map $\tilde{f}: K \rightarrow C(Z,C)$ such that $f = \mu \circ (a \otimes_A \tilde{f})$: $\varphi \otimes_A K \rightarrow C(F,Z) \otimes_A C(Z,C) \rightarrow C(F,C)$. For existence, we let $\tilde{f}$ be the composite

$$K \cong A \otimes_A K \xrightarrow{\eta \otimes A 1} \psi \otimes_B \varphi \otimes_A K \xrightarrow{b \otimes \varphi \tilde{f}} C(Z,F) \otimes_B C(F,C) \xrightarrow{\mu} C(Z,C);$$

(5)

now rewriting with the right-hand square of (2) and using the triangle identities and $F$’s preservation of units shows that $f = \mu \circ (a \otimes_A \tilde{f})$. For uniqueness, let $g: K \rightarrow C(Z,C)$ also satisfy $f = \mu \circ (a \otimes_A g)$. Substituting into (5) shows that $\tilde{f}$ is the composite

$$K \cong A \otimes_A K \xrightarrow{\eta \otimes A 1} \psi \otimes_B \varphi \otimes_A K \xrightarrow{b \otimes \varphi a \otimes A 1} C(Z,F) \otimes_B C(F,Z) \otimes_A C(Z,C) \xrightarrow{\mu} C(Z,C);$$

which by rewriting with the left square of (2) and using $Z$’s preservation of identities is equal to $g$. This proves the equivalence (a) $\Leftrightarrow$ (c); now (b) $\Leftrightarrow$ (c) follows by duality.

1.3. Examples. We first consider examples wherein $\mathcal{W}$ is the one-object bicategory corresponding to a monoidal category $\mathcal{V}$.

- Let $\mathcal{V} = \text{Set}$, and let $\varphi$ be the weight for splittings of idempotents. The result recovers the bijection, for an idempotent $e: A \rightarrow A$, between: maps $p: A \rightarrow B$ coequalising $e$ and $1_A$; maps $i: B \rightarrow A$ equalising $e$ and $1_A$; and pairs $(i,p)$ with $pi = 1_A$ and $ip = e$.

- Let $\mathcal{V} = \text{Set}_*$, and let $\varphi$ be the weight for an initial object. The result recovers the bijection in a pointed category between: initial objects; terminal objects; and objects $X$ with $1_X = 0_X$. 
• Let $\mathcal{V} = \textbf{Ab}$, and let $\varphi$ be the weight for binary coproducts. The result recovers the bijection, for objects $A,B$ in a pre-additive category, between: coproduct diagrams $i_1: A \to Z \leftarrow B$: $i_2$; product diagrams $p_1: A \leftarrow Z \to B$: $p_2$; and tuples $(i_1,i_2,p_1,p_2)$ such that $p_j i_k = \delta_{jk}$ and $i_1 p_1 + i_2 p_2 = 1_Z$.

• Let $\mathcal{V} = \mathbf{-Lat}$, and let $\varphi$ be the weight for $J$-fold coproducts (for $J$ a small set). The result recovers the bijection, for objects $(A_j : j \in J)$ in a sup-lattice enriched category, between: coproduct diagrams $(i_j : A_j \to Z)_{j \in J}$; product diagrams $(p_j : Z \to A_j)_{j \in J}$; and families $(i_j)_{j \in J}$ and $(p_j)_{j \in J}$ with $p_j i_k = \delta_{jk}$ and $\bigvee_j i_j p_j = 1_Z$.

• Let $\mathcal{V} = k\text{-Vect}$ for $k$ a field of characteristic zero, let $G$ be a finite group, and let $\varphi: k \to kG$ be the trivial right $kG$-module $k$. By Burnside’s Lemma, $\varphi$ has a right adjoint $kG \to k$ given by the trivial left $kG$-module $k$. Now the result recovers the bijection, for a $G$-representation $A$ in a $k$-linear category, between: maps $p: A \to Z$ exhibiting $Z$ as an object of coinvariants of $A$; maps $i: Z \to A$ exhibiting $Z$ as an object of invariants of $A$; and pairs of maps $(i,p)$ with $pi = 1_Z$ and $ip = \frac{1}{|G|} \sum_{g \in G} g$.

We conclude with two examples where $\mathcal{W}$ is a genuine bicategory.

• Let $(\mathcal{C}, j)$ be a subcanonical site, and let $\mathcal{W}$ denote the full sub-bicategory of $\text{Span}(\text{Sh}(\mathcal{C}))^{\text{op}}$ on objects of the form $\mathcal{C}(\cdot, X)$. To any prestack $p: \mathcal{E} \to \mathcal{C}$ over $\mathcal{C}$, we may (as in [1]) associate a $\mathcal{W}$-category with objects those of $\mathcal{E}$, extents $\epsilon(a) = p(a)$, and hom-objects from $a$ to $b$ given by the span $\mathcal{C}(-,pa) \leftarrow \mathcal{E}(a,b) \to \mathcal{C}(-,pb)$ in $\text{Sh}(\mathcal{C})$; here $\mathcal{E}(a,b)(x)$ is the set of all triples $(f,g,\theta)$ with $f: pa \leftarrow x \to pb$: $g$ in $\mathcal{C}$ and $\theta: f^*(a) \to g^*(b)$ in $\mathcal{E}_x$ (note that $\mathcal{E}(a,b)$ is a sheaf by the prestack condition).

For any cover $(f_i: U_i \to U)_{i \in I}$ in $\mathcal{C}$, we have a $\mathcal{W}$-category $R[f]$ with object set $I$, extents $\epsilon(i) = U_i$ and hom-objects $R[f](j,i) = \mathcal{C}(-,U_j) \leftarrow \mathcal{C}(-,U_i \times_U U_j) \to \mathcal{C}(-,U_i)$. There is a profunctor $\varphi: U \to R[f]$ with components given by the spans $\varphi(i,\ast) = \mathcal{C}(-,U_i) \leftarrow \mathcal{C}(-,U_i \times_U U_j) \to \mathcal{C}(-,U_i)$. Writing $\psi: R[f] \to U$ for the reverse profunctor, it is not hard to see that $\varphi \dashv \psi$ (in fact they are adjoint pseudoinverse).

The result now says the following. Given a prestack $p: \mathcal{E} \to \mathcal{C}$, a cover $(f_i: U_i \to U)$ in $\mathcal{C}$, and a family of spans $p_{ij}: a_i \leftarrow a_{ij} \to a_j: q_{ij}$ in $\mathcal{E}$ whose legs are cartesian over the projections $U_i \leftarrow U_i \times_U U_j \to U_j$, there is a bijection between: cocones $(h_i: a_i \to a)$ in $\mathcal{E}$ over the $f_i$’s that are colimiting for the diagram comprised of the $p_{ij}$’s and $q_{ij}$’s; universal objects $a \in \mathcal{E}_U$ equipped with vertical maps $f_i^*(a) \to a_i$ fitting into double pullback squares

$$
\begin{array}{ccc}
\bullet & \to & \bullet \\
| & & | \\
a_i & \leftarrow & a_{ij} \to a_j \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\bullet & \leftarrow & \bullet
\end{array}
$$

and objects $a \in \mathcal{E}_U$ equipped with a family of maps $(h_i: a_i \to a)$ cartesian over the $f_i$’s. This generalises [6, Proposition 5.2(b)]$^1$.

$^1$The proposition numbering here is taken from the TAC reprint.
• Let $\mathcal{W}$ denote the bicategory whose objects are sets, and whose hom-category $\mathcal{W}(X,Y)$ is the category of finitary functors $\textbf{Set}/Y \to \textbf{Set}/X$; note that $\mathcal{W}(X,Y) \simeq [\text{Fam}(Y) \times X, \textbf{Set}]$, where $\text{Fam}(Y)$ has as objects, finite lists of elements of $Y$, and as maps $(y_0,\ldots,y_m) \to (z_0,\ldots,z_n)$, functions $f: [m] \to [n]$ such that $y_i = z_{f(i)}$. To any cartesian multicategory $M$ (i.e., a Gentzen multicategory in the sense of [3]) we may associate a $\mathcal{W}$-category $\mathcal{M}$ whose objects of extent $X$ are $X$-indexed families of objects of $M$, and whose hom-object between families $(a_x)_{x \in X}$ and $(b_y)_{y \in Y}$ is the presheaf

$$\mathcal{M}((b_y),(a_x))(y_0,\ldots,y_m; x) = M(b_{y_0},\ldots,b_{y_m};a_x)$$

in $[\text{Fam}(Y) \times X, \textbf{Set}]$; reindexing along maps in $Y$ makes use of the cartesianness of the multicategory structure. Composition and units in $\mathcal{M}$ follow from those in $M$.

Given a finite set $X = \{x_0,\ldots,x_n\}$, let $\varphi: 1 \to X$ be the $\mathcal{W}$-profunctor whose unique component is the representable $y(x_0,\ldots,x_n;\ast) \in [\text{Fam}(X) \times 1, \textbf{Set}]$. This has a right adjoint $\psi: X \to 1$ whose unique component is the presheaf $\Sigma_{x \in X}y(\ast; x) \in [\text{Fam}(1) \times X, \textbf{Set}]$. The result now establishes a bijection, for any finite family $(a_0,\ldots,a_n)$ of objects in a cartesian multicategory $M$, between data of the following three forms: first, an object $a$ and a multimap $i \in M(a_0,\ldots,a_n;a)$, composition with which induces bijections between $M(b_0,\ldots,b_k,a,c_0,\ldots,c_l;d)$ and $M(b_0,\ldots,b_k,a_0,\ldots,a_n,c_0,\ldots,c_l;d)$; second, an object $a$ and unary maps $p_j \in M(a;a_j)$, composition with which establishes bijections between $M(b_0,\ldots,b_k;a)$ and $\Pi_jM(b_0,\ldots,b_k;a_j)$; third, an object $a$ and maps $i$ and $p_j$ as above such that $p_j \circ i = \pi_j \in M(a_0,\ldots,a_n;a_j)$ and $i \circ (p_0,\ldots,p_n) = 1_a \in M(a;a)$. This generalises [2, Proposition 3.5].

References


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