VALUATION OF FINANCIAL DERIVATIVES UNDER REGIME SWITCHING MODELS

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Signed Statement

I, Kun Fan, declare that this thesis, submitted in fulfilment of the requirements for the award of Doctor of Philosophy in Applied Finance and Actuarial Studies at Macquarie University and Probability and Mathematical Statistics at East China Normal University, wholly represents my own work unless otherwise referenced or acknowledged. The document has not been previously included in a thesis, dissertation or report submitted to this university or any other institution for a degree, diploma or other qualifications.

The results in Chapter 2 have been published on Economic Modelling (2014; 37, 296-305). Other chapters are unpublished papers. I finished the works independently with necessary directions from my supervisors, Tak Kuen Siu, Xian Zhou and Rongming Wang. I thank Yang Shen for his assistance.

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Dedication

To my parents, Shiyan Fan and Xiaoyun Li
Abstract

Following the global financial crisis of 2008, the impacts of changes in (macro)-economic conditions and business cycles have attracted increasing interests in the new millennium. Regime-switching models have been considered as a natural tool of pricing financial derivatives by both academic researchers and industrial practitioners since Hamilton (1989) introduced this class of models into financial econometrics. Regime-switching models typically use the states of the modulating Markov chain to represent the states of an economy, depicted by some (macro)-economic indicators. By adopting this methodology, regime-switching models can incorporate the impacts of structural changes in (macro)-economic conditions. Consequently, it is practical to consider the valuation of financial derivatives under regime-switching models.

In this thesis, the Markov chain we adopt is a continuous-time and finite-state Markov chain, either observable or unobservable, under regime-switching models. Our modelling setup includes regime-switching diffusion models, regime-switching jump-diffusion models, regime-switching stochastic interest rate models, double regime-switching models and hidden Markov models. Under regime-switching models, the additional uncertainty leads to an incomplete financial market. The selection of a pricing kernel in an incomplete market has long been discussed as more than one equivalent martingale measures may exist in an incomplete market. In this thesis, both the regime-switching
Esscher transform and the minimal martingale measure approach are considered in selecting a pricing kernel under regime-switching models. To obtain analytical pricing formulae for the financial derivatives, we mainly focus on the applications of the fast Fourier transform under regime-switching models.

In Chapter 1, we first briefly introduce the pricing models of financial derivatives. We further provide a literature review on topics including option valuation under different kinds of regime-switching models, hidden Markov models, stochastic interest rate models, approaches to determine an equivalent martingale measure in incomplete markets, and fast Fourier transform. Then the mathematical tools to be used in this thesis are introduced. More specifically, we give the mathematical representation of the modulating Markov chain and describe how to obtain analytical pricing formulae via Fourier transform and discretize the pricing formulas via the fast Fourier transform method. Finally, we give an overview of the papers to be included in this thesis.

In Chapters 2-4, we consider the valuations of various options under different regime-switching models where model parameters are modulated by a continuous-time, finite-state and observable Markov chain. Analytical pricing formulae for these options are obtained via the inverse Fourier transform and calculated via the fast Fourier transform, providing an easier and neater way to calculate the option prices.

In Chapter 2, we consider the valuation of foreign equity options, settling in one currency while the underlying assets are denominated in a different currency, under a Markovian regime-switching mean-reversion lognormal model. Intuitively, the valuation of the so-called foreign equity options has to deal with the joint dynamics of both the foreign equity and the exchange rate. Chapter 2 considers two kinds of foreign equity options, one with a strike price in the foreign currency ($FEO_F$) and the other with a strike price in the domestic currency ($FEO_D$), under the assumption that the foreign exchange rate follows a mean-reversion lognormal model with regime-switching. To u-
tilize the fast Fourier transform method, the characteristic function of the logarithmic entity price is needed. For $FEO_F$, a measure change technique has to be applied first to take the expectation of the foreign exchange rate as the numéraire. For $FEO_D$, a simple summation of the characteristic functions of the logarithmic foreign equity price and the logarithmic foreign exchange rate are calculated. Then we derive analytical pricing formulae for both $FEO_F$ options and $FEO_D$ options.

The valuation of power options are discussed in Chapter 3 by considering a Markovian regime-switching jump-diffusion model, where a Poisson random measure is adopted to depict the jump component. Power options provide investors with a choice of financial products with nonlinear payoff functions. This feature is attractive to many investors, especially in a financial market that involves many types of nonlinear risks. A unique equivalent martingale measure is selected by adopting a version of regime-switching Esscher transform. Then, under the risk-neutral probability measure, a standard application of the inverse Fourier transform is applied to obtain analytical pricing formulae for the power option. In the numerical analysis, particular parametric forms of the compensator measure, including the Markov-modulated inverse Gaussian process and the Markov-modulated Merton jump-diffusion model, are considered.

Chapter 4 discusses the option valuation under a regime-switching stochastic interest rate model, which may increase the long-term effectiveness of the model. We start with a risk-neutral probability measure. To take the zero-coupon bond value as the numéraire, a measure change technique is applied to change the risk-neutral probability measure into a forward measure. By deriving the formulae for the characteristic function of the logarithmic return of the underlying asset, the fast Fourier transform can then be applied.

Chapters 5-6 consider the valuation of insurance products with embedded-option features under regime-switching models. In Chapter 5, we investigate the valuation
of equity-linked annuities with mortality risk under a double regime-switching model, where the modulating Markov chain is a continuous-time, finite-state, observable chain. In addition to the assumption that model parameters will change when the states of the Markov chain switch, we also assume that a jump of the price level of the underlying investment fund will be triggered when a state transition of the chain occurs. One of the main features of the double regime-switching models is to provide an endogenous way to determine the regime-switching risk. To specify a unique pricing kernel, we employ both the generalized version of regime-switching Esscher transform and the minimal martingale method to determine a unique equivalent martingale measure, respectively. Note that the transition matrix of the Markov chain will also change. There have been many works adopting the option valuation techniques to price insurance policies with embedded-option features. In Chapter 5, we write the payoff of the equity-linked annuities as a combination of the payoffs of several European-style options. Consequently, we can utilize option valuation techniques to price the equity-linked annuities. Here, the technique of the fast Fourier transform applied to option valuation can be used to price such equity-linked annuities. Numerical analysis and sensitivity analysis provide us with intuitive understandings of the valuation of equity-linked annuities.

Chapter 6 discusses the valuation of dynamic fund protection under a hidden Markov model, where the states of the modulating Markov chain are unobservable. To protect investors from downside risk, a dynamic fund protection plan is an effective and convenient tool. The payoff function of a dynamic fund protection plan can be written as a product of the payoff of a fixed strike lookback call and the exchange rate. Consequently, option valuation techniques can also be utilized to price dynamic fund protection plans. In this chapter, the approach of partial differential equations is adopted to price dynamic fund protection plans by a three-stage estimation method. It consists of the Baum-Welch algorithm to estimate the model parameters of the hidden
Markov chain, the Viterbi algorithm to select the most-probable path for the chain, and the maximum likelihood method to estimate the model parameters. The three-stage estimation method is intuitively appealing and easy to implement in practice.
Chapter 1

Introduction

1.1 Overview

Derivatives, together with equities and debts, have been considered as the three most important financial instruments for centuries. The term “derivative” here refers to a contract with the value derived from an underlying entity. The valuation of derivatives has been a long-lasting issue. As we know, there are numerous kinds of derivatives in both exchange-trading and over-the-counter markets. Amongst the different kinds of derivatives, options play a vital role in the financial market. According to Wikipedia, the definition of an option is as follows:

“An option is a contract which gives the buyer (the owner) the right, but not the obligation, to buy or sell an underlying asset or instrument at a specified strike price on or before a specified date.” (Wikipedia)

There are many forms of options, among them call options and put options are two
fundamental types. Generally, options are classified into three categories, European-style options, American-style options and exotic options. The former two options are also called vanilla options. From another line, options can also be classified according to the types of the underlying entities, such as stocks, stock index, equities, commodities, etc. Options and financial products with embedded-option features have become so important that we can hardly find an investment portfolio without these products. Consequently, the valuation of such financial derivatives deserves more attention.

Option valuation has played a vital role in the development of modern finance, especially since the introduction of the seminal works of Black and Scholes (1973) and Merton (1973). The valuation of options has long been a theoretically and practically important topic in the area of finance. The path-breaking works have gained long-lasting popularity due to the easy-implementation of the closed-form option pricing formulae. However, numerous empirical studies have revealed that the Black-Scholes-Merton model lacks the ability to describe some vital stylized features of the underlying assets. In order to overcome the shortcomings and improve the efficiency of the Black-Scholes-Merton model, both academic researchers and industry practitioners have dedicated to extending the Black-Scholes-Merton model in various possible directions, including jump-diffusion models (Merton (1976)), stochastic volatility models (Hull and White, 1987; Wiggins, 1987; Heston, 1993; etc.), regime-switching models, etc.

1.2 Literature review

Regime-switching models

Compared with other extensions, the class of regime-switching models has its own unique advantages. Specifically, regime-switching models have the capability to incor-
porate the changes of the model dynamics brought by the changing (macro)-economic conditions. Consequently, regime-switching models have attracted considerable interests and been applied to various financial areas. The history of regime-switching models can be traced back to the works of Quandt (1958) and Goldfeld and Quandt (1973). The applications of regime-switching models were popularized by Hamilton (1989) to econometrics. Since then, regime-switching models have gained unique preferences of researchers when they consider financial problems, including asset allocation, option valuation, etc. Following the terminology in Shen et al. (2013), regime-switching models can be roughly classified into two categories. If a regime-switching model assumes that model parameters will change when the states of the Markov chain switch, it is called the single regime-switching model (see Bollen, 1998; Guo, 2001; Buffington and Elliott, 2002; Duan et al., 2002; Elliott et al., 2005; Siu, 2005; Elliott and Osakwe, 2006; Elliott et al., 2007; Boyle and Draviam, 2007; Siu, 2008; and others). If in addition, a jump of the price level of the underlying asset is triggered when a switch of regimes occurs, then the regime-switching model is referred to as the double regime-switching model. Examples of option valuation under double regime-switching models include Naik (1993), Yuen and Yang (2009), Elliott and Siu (2011), Shen et al. (2013) and others.

**Hidden Markov models**

Although previous works have been justified empirically, the assumption that the states of the modulating Markov chain are observable may be violated. In practice, it is generally difficult to observe the states of the underlying economy. Consequently, the valuation of financial derivatives under a hidden (unobservable) Markov model has its own practical value. An overview of hidden Markov models can be referred to Elliott et al. (1994) and Ching et al. (2013). There are many approaches to
estimate the model parameters. The expectation-maximization (EM) algorithm and the traditional maximum likelihood estimation are two typical approaches. To apply the EM algorithm, many filters and smoothers for the hidden Markov chain have been proposed, including exact adaptive filters by Elliott (1994) and finite-dimensional filters by Elliott and Krishnamurthy (1997). These filters and smoothers are mainly based on the extensions of the Kalman filter and the Wonham filter. Elliott and Malcolm (2000), Elliott et al. (2003) and Malcolm and Elliott (2010) proposed new EM algorithms to deal with the stochastic integrations in the dynamic of the logarithmic return of the underlying asset introduced by the modulated volatility. A two-stage estimation method, involving the Viterbi algorithm to estimate the hidden Markov chain and the maximum likelihood method to estimate model parameters, was adopted by Siu et al. (2005) and Elliott et al. (2012).

Compared with the works investigating the valuation of derivatives under a regime-switching model with an observable Markov chain, relatively little attention has been paid to the valuation of derivatives under a hidden Markov model, where the states of the modulating Markov chain may not be observable. Landén (2000) considered the valuation of a bond under the assumption that the appreciation rate and the volatility of the short rate are modulated by a hidden Markov chain. Ishijima and Kihara (2005) estimated the parameters of the hidden Markov chain via the Baum-Welch algorithm. Then analytical formulae for European-style options were obtained. Liew and Siu (2010) discussed the valuation of European-style options under a regime-switching model with a discrete-time hidden Markov model. Both Esscher transform and the extended Girsanov’s principle were considered to select an equivalent martingale measure. They adopted an EM algorithm based on a recursive filter to estimate the model parameters. Elliott and Siu (2013) considered the option valuation under a pure-jump model modulated by a continuous-time finite-state hidden Markov model, where the
compensator of the random pure-jump process is modulated by the modulating Markov chain. A robust filter was applied to estimate the parameters of the Markov chain and the model dynamics.

**Stochastic interest rate models**

Different stochastic interest rate models have been proposed to overcome the disadvantage of the constant interest rate assumption under the Black-Scholes-Merton model. Some popular stochastic interest rate models include those proposed by Vasicek (1977), Cox et al. (1985), Hull and White (1990), among others. One common feature of these models is the mean-reverting property of the interest rate. The short-term effectiveness of these models were justified by many empirical studies. Due to the advantages of regime-switching models, it is reasonable to expect that regime-switching stochastic interest rate models may improve the long-term effectiveness of the existing stochastic interest rate models. Examples of regime-switching stochastic interest rate diffusion models can be found in Elliott and Mamon (2003), Elliott and Wilson (2007), Elliott and Siu (2009) and Elliott et al. (2011a). By adopting the method of stochastic flows, Siu (2010) considered the valuation of a bond under a jump-augmented Vasicek model. A partial differential equation approach was applied in Shen and Siu (2013a) to obtain an exponential affine formula for a zero-coupon bond.

**Equivalent martingale measure in incomplete markets**

The additional uncertainty brought by the Markov chain will result in an incomplete market. The selection of a pricing kernel in an incomplete market is an important issue in option valuation. The relationship between the arbitrage theory and the existence of an equivalent martingale measure was investigated in Harrison and Kreps (1979) and Harrison and Pliska (1981). Since there exist more than one equivalent martingale measures in an incomplete market, researchers have proposed many approaches
to select an equivalent martingale measure in an incomplete market. (Locally) risk-minimization method was applied to identify a unique equivalent martingale measure, referred to as the *minimal martingale measure*, by Föllmer and Sondermann (1986), Föllmer and Schweizer (1991) and Schweizer (1996). The notion of minimal martingale entropy measure was proposed and further discussed in Miyahara (2001) and Frittelli (2000). Since then, the minimal martingale entropy measure has been widely applied in the valuation of financial derivatives. Arai (2001) discussed the relationship between the minimal martingale measure and the minimal martingale entropy measure. Among different approaches to determine an equivalent martingale measure, Esscher transform has become a time-honored tool since Gerber and Shiu (1994) creatively applied it to the valuation of financial derivatives. The seminal work provides a comparatively easy way to overcome the difficulty in determining a pricing kernel. One of the main features of Esscher transform is to allow the flexibility to deal with different dynamics of the underlying entity. Bühlmann et al. (1996) and Kallsen and Shiryaev (2002) extended the work of Gerber and Shiu (1994) to a general semimartingale. Elliott et al. (2005) introduced a regime-switching Esscher transform to investigate the option valuation under regime-switching models, which was further justified by Siu (2008, 2011). The selection of an equivalent martingale measure via the regime-switching Esscher transform was justified by the minimal entropy martingale measure. Furthermore, it is well documented that this method can be justified by the maximization of the expected power utility function. The regime-switching Esscher transform have been adopted to select an equivalent martingale measure under different regime-switching models. Examples include Elliott et al. (2005), Siu (2005), Elliott et al. (2007), Ching et al. (2007), Siu (2008), Siu et al. (2008a), Lin et al. (2009), Siu and Yang (2009), Ng and Li (2011), Shen et al. (2013), Elliott and Siu (2013), etc. How to determine an equivalent martingale measure via an extended Girsanov’s principle was discussed in Elliott and
Madan (1998). Comparisons between Esscher transform and the extended Girsanov’s principle were considered in Badescu et al. (2009) and Liew and Siu (2010). Shen and Siu (2013b) compared three approaches to choose an equivalent martingale measure, including the game theoretic approach, Esscher transform and the general equilibrium approach.

**Fast Fourier transform**

Fourier transform has been widely applied to the valuation of financial derivatives. The faster calculation speed of the discrete Fourier transform may be one of the main reasons why the fast Fourier transform (FFT) method attracts so much attention from both academics and industry. Carr and Madan (1999) made an early attempt to apply the FFT method to price European-style options under the variance gamma (VG) model. Since then, the FFT method has been applied to the valuation of options under different models. For example, Benhamou (2002) discussed the valuation of discrete Asian options in non-lognormal density cases. Dempster and Hong (2002) presented a two-dimensional FFT and considered the valuation of spread options under a three-factor stochastic volatility model. Černý (2004) discussed applications of the FFT in finance. By adopting the FFT technique, Liu et al. (2006) investigated the valuation of options under a regime-switching model and Wong and Guan (2011) considered the valuation of American options under a Lévy process.

Although the FFT approach is significantly faster than other numerical methods, such as finite difference method and Monte Carlo simulation, it still has approximation errors when we adopt a discrete sum to approximate the integral. To control the approximation errors, Carr and Madan (1999) discussed the selection of the upper limit of the integral and gave a sufficient condition to guarantee the square integrability property of the dampened pricing formulae. Numerical errors in discretizing the pricing
formula were discussed in Lee (2004). Liu et al. (2006) also showed that the errors are small. Kwok et al. (2012) discussed the effectiveness of FFT in option valuation and suggested $\alpha = 3$ be adopted from the perspective of numerical illustrations.

1.3 Preliminaries

This subsection presents a brief introduction of the basic definitions and tools to be used in the subsequent chapters. The contents in this subsection are mainly based on Elliott (1982), Elliott et al. (1994), Dufour and Elliott (1999), Carr and Madan (1999), James (2002, 2005), Yin et al. (2005) and Siu et al. (2008a).

Markov chain

Let $\mathcal{T} := [0, T]$ be the time horizon ($T < \infty$). Define $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t \in \mathcal{T}}, \mathcal{P})$ to be a filtered complete probability space, where $(\mathcal{F}(t))_{t \in \mathcal{T}}$ is a right continuous $\mathcal{P}$-complete filtration. Let $X := \{X(t)|t \in \mathcal{T}\}$ denote a continuous-time, finite-state Markov chain defined on $(\Omega, \mathcal{F}, \mathcal{P})$ with a finite state space $S$. Following Elliott et al. (1994), without loss of generality, the state space of the Markov chain can be identified as a finite set of unit vectors $E := \{e_1, e_2, \ldots, e_N\}$, where $e_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^N$ with 1 at its $i$th position and 0 elsewhere.

Let $\mathcal{F}^X(t)$ denote the filtration generated by the Markov chain $X$. By adopting the canonical state space $E$, Elliott et al. (1994) obtained the following semi-martingale representation of the Markov chain:

$$X(t) = X(0) + \int_0^t A X(s) ds + M(t), \quad (1.3.1)$$

where $A := [a_{ij}]_{N \times N}$ is the generator of the Markov chain and $\{M(t)|t \in \mathcal{T}\}$ is a right continuous martingale with respect to $\mathcal{F}^X(t)$. 

8
Itô's differentiation rule

The Itô’s differentiation rule for a Markov-modulated process will be used in the following chapters. Following James (2002, 2005), Yin et al. (2005) and Siu et al. (2008a), we present the Itô’s differentiation rule for a Markov-modulated jump-diffusion process. Let $y$ be the transpose of a matrix or a vector and $R_0 := \mathbb{R} \setminus \{0\}$. Denote $\mathcal{B}(T)$ and $\mathcal{B}(R_0)$ be the Borel $\sigma$-fields generated by open subsets of $T$ and $R_0$, respectively. Suppose $\gamma(dt,dz)$ is a Poisson random measure defined on the product space $(T \times R_0, \mathcal{B}(T) \otimes \mathcal{B}(R_0))$. Denote $\nu_{X(t)}(dt,dz)$ and $\tilde{\gamma}(dt,dz)$ as the compensator and the compensated version of the random measure $\gamma(dt,dz)$, respectively.

$$
\tilde{\gamma}(dt,dz) = \gamma(dt,dz) - \nu_{X(t)}(dt,dz).
$$

Furthermore, assume the compensator $\nu_{X(t)}(dt,dz)$ is defined as follows:

$$
\nu_{X(t)}(dt,dz) := \omega_{X(t)}(dz)dt = \sum_{j=1}^{N} \langle X(t), e_j \rangle \omega_j(dz)dt,
$$

where $\omega_j(dz)$ is a Lévy measure on the space $R_0$ when the Markov chain is in the $j$-th state. Then the Markov-modulated kernel-biased completely random measure can be defined as

$$
\chi_{X(t)}(dt) := \int_{R_0} h(t,X(t),z)\gamma(dt,dz),
$$

Here, the selection of $\omega$, $\eta$ and $h$ has to ensure the existence of the kernel-biased completely random measure.

Suppose the dynamic of the Markov-modulated jump-diffusion process $Y := \{Y(t) | t \in T\}$ is given by:

$$
Y(t) = Y(0) + \int_0^t \mu(s,X(s))ds + \int_0^t \sigma(s,X(s))dW(s) \int_0^t \int_{R_0} h(s,X(s),z)\tilde{\gamma}(ds,dz),
$$

where $\langle \cdot, \cdot \rangle$ is the inner product.
Let $g(\cdot, \cdot, \cdot) : \mathcal{T} \times \mathcal{R} \times \mathcal{E} \to \mathcal{R}$ be a function such that $g(t, y, e_i)$ is a sufficiently smoothed function on $\mathcal{T} \times \mathcal{R} \times \mathcal{E}$. Note that $g_y$, $g_t$ and $g_{yy}$ represent the first derivative of $g$ with respect to $y$, $t$ and the second derivative with respect to $y$ respectively. Define an operator $\mathcal{L}$ as follows

$$
\mathcal{L} g(t, y, e_i) = g_t(t, y, e_i) + g_y(t, y, e_i) \mu(t, e_i) + \frac{1}{2} \text{tr} [g_{yy}(t, y, e_i) \sigma \sigma'(t, e_i)] + \int_{\mathcal{E}_0} [g(t, y + h(t, e_i, z), e_i) - g(t, y, e_i) - g_y(t, y, e_i) h(t, e_i, z)] \omega_{e_i}(dz) + \langle g(t, y), A(t) e_i \rangle,
$$

where $g(t, y) = (g(t, y, e_1), \ldots, g(t, y, e_N))$ and $\langle g(t, y), A(t) e_i \rangle = \sum_{j=1}^{N} a_{ij} g(t, y, e_j)$. Then, the Itô’s differentiation rule is given by

$$
g(t, Y(t), X(t)) - g(0, Y(0), X(0)) = \int_{0}^{t} \mathcal{L} g(s, Y(s), X(s)) ds + \int_{0}^{t} g_y(s, Y(s), X(s)) \sigma(s, X(s)) dW(s) + \int_{0}^{t} \int_{\mathcal{E}_0} [g(s, Y(s) + h(z), X(s)) - g(s, Y(s), X(s))] \gamma(ds, dz)
$$

$$
+ \int_{0}^{t} \langle g(t, Y(t)), dM(t) \rangle.
$$

The Itô’s differentiation rule presented here is a special case of Itô’s differentiation rule for semimartingales in $\mathcal{R}^N$, (see, for example, Elliott (1982) and Jacod and Shiryaev (1987)). Note that if there is no jump component, Eq. (1.3.2) becomes the Itô’s differentiation rule for a Markov-modulated diffusion process.

**Girsanov’s theorem for the Markov chain**

Here we focus on a measure change for the Markov chain $X$ considered in Dufour and Elliott (1999). The semi-martingale representation of the Markov chain under the probability measure $\mathcal{P}$ is given in Equation (1.3.1). The objective is to introduce a
new martingale measure which is equivalent to \( \mathcal{P} \) such that \( X \) is a Markov chain with a second rate matrix \( B := [b_{ij}]_{N \times N} \) satisfying \( b_{ij} \geq 0 \) for \( i \neq j \) and \( \sum_{i=1}^{N} b_{ij}(t) = 0 \) under the new probability measure.

Write \( N := \{N(t)|t \in T\} \) as a vector-valued counting process, with component \( N_i(t) \) representing the number of jumps to state \( e_i \) made by Markov chain \( X \) up to time \( t \). Define

\[
\mathbf{a} := (a_{11}, a_{22}, \ldots, a_{NN})',
\]

\[
A_0 := A - \text{diag}(\mathbf{a}),
\]

\[
D := [d_{ij}]_{N \times N} = [b_{ij}/a_{ij}]_{N \times N},
\]

\[
d := (d_{11}, d_{22}, \ldots, d_{NN})',
\]

\[
D_0 := D - \text{diag}(d).
\]

Consider the process \( \Lambda := \{\Lambda(t)|t \in T\} \) on \( (\Omega, \mathcal{F}, \mathcal{P}) \) given by

\[
\Lambda(t) = 1 + \int_0^t \Lambda(s-) [D_0 X(s-) - \mathbf{1}'] (dN(s) - A_0 X(s-) ds),
\]

where \( \mathbf{1} := (1, 1, \ldots, 1)' \in \mathbb{R}^N \). It is easy to see that \( \Lambda \) is an \( (\mathcal{F}^X, \mathcal{P}) \)-martingale.

Define a new probability measure \( \mathcal{Q} \) by

\[
\left. \frac{d\mathcal{Q}}{d\mathcal{P}} \right|_{\mathcal{F}^X(T)} := \Lambda(T).
\]

Then under \( \mathcal{Q} \), the intensity matrix of the Markov chain is \( B \) and the semi-martingale dynamic of the chain \( X \) is as follows:

\[
X(t) = X(0) + \int_0^t B X(s) ds + M^B(t),
\]

where \( M^B := \{M^B(t)|t \in T\} \) is an \( \mathbb{R}^N \)-valued \( (\mathcal{F}^X, \mathcal{Q}) \)-martingale.

**Fast Fourier transform**
Fast Fourier transform was introduced in Carr and Madan (1999). For the sake of completeness, we present a brief introduction of the application of fast Fourier transform. Define by $S(T)$ the value of the underlying asset at the maturity time and $K$ the strike price. Suppose that the price of a $T$-maturity European-style call option at time 0 is given by

$$C(0, T, K) = E\left[ \exp \left( -\int_0^T r(t) dt \right) (S(T) - K)_+ \right],$$

(1.3.3)

where $E[\cdot]$ denote the expectation under the risk-neutral probability measure.

Let $s(T) := \ln S(T)$ and $k := \ln K$ denote the logarithmic of the asset price at time $T$ and the strike value, respectively. Then

$$C(0, T, k) = E\left[ \exp \left( -\int_0^T r(t) dt \right) (e^{s(T)} - e^k)_+ \right],$$

and the dampened call price is defined as

$$c(0, T, k) := \exp(\alpha k) C(0, T, k),$$

where $\alpha$ is selected to ensure that $c(0, T, k)$ is square integrable in $k$. Then the Fourier transform of the dampened call option price is given by

$$\xi(0, T, u) = \int_{\mathbb{R}} e^{iku} c(0, T, k) dk.$$ 

The pricing formulae for the European-style call option can be obtained via the inverse Fourier transform as follows:

$$C(0, T, k) = \frac{e^{-\alpha k}}{2\pi} \int_{\mathbb{R}} e^{-iku} \xi(0, T, u) du = \frac{e^{-\alpha k}}{\pi} \int_0^\infty e^{-iku} \xi(0, T, u) du.$$ 

(1.3.4)

To derive the Fourier transform of the dampened call option price, one standard way is to utilize the relationship between the Fourier transform of the dampened call option price and the characteristic function of the logarithmic asset price. In this thesis, under
regime-switching models, the conditional characteristic function of the logarithmic asset price given $\mathcal{F}^X(T)$ has to be derived first. For each $t \in T$ and $u \in \mathbb{R}$, let

$$\zeta_{s(t)|\mathcal{F}^X(t)}(0, t, u) := \exp \left( - \int_0^t r(s)ds \right) \varphi_{s(t)|\mathcal{F}^X(t)}(0, t, u)$$

and $\zeta_{s(t)}(0, t, u) = E[\zeta_{s(t)|\mathcal{F}^X(t)}(0, t, u)]$ be the discounted versions of conditional characteristic function of $s(t)$ given $\mathcal{F}^X(t)$ and the unconditional function of $s(t)$ under the risk-neutral probability measure, respectively. Let $F_{s(T)|\mathcal{F}^X(T)}(ds)$ denote the conditional density function of $s(T)$ given $\mathcal{F}^X(T)$ and $R_T = \int_0^T r(t)dt$. Then

$$\xi(0, T, u) = \int_{\mathbb{R}} e^{iku} c(0, T, k)dk$$

$$= \int_{\mathbb{R}} e^{iku} e^{ak} E \left[ \exp \left( - \int_0^T r(t)dt \right) (S(T) - e^k)^+ \right] dk$$

$$= E \left[ \int_{\mathbb{R}} e^{iku} e^{ak} E[e^{-RT} (S(T) - e^k)^+ | \mathcal{F}^X(T)]dk \right]$$

$$= E \left[ \int_{\mathbb{R}} e^{-RT} e^{iku} e^{ak} \int_k^\infty (e^s - e^k) F_{s(T)|\mathcal{F}^X(T)}(ds)dk \right]$$

$$= E \left[ \int_{\mathbb{R}} e^{-RT} F_{s(T)|\mathcal{F}^X(T)}(ds) \int_{-\infty}^s (e^{(\alpha+iu)k} - e^{(1+\alpha+iu)k})dk \right]$$

$$= E \left[ \int_{\mathbb{R}} e^{-RT} F_{s(T)|\mathcal{F}^X(T)}(ds) \left( \frac{e^{(1+\alpha+iu)s}}{\alpha + iu} - \frac{e^{(1+\alpha+iu)s}}{1 + \alpha + iu} \right) \right]$$

$$= \frac{\zeta(0, T, u - i(\alpha + 1))}{\alpha^2 + \alpha - u^2 + i(2\alpha + 1)u}.$$ 

Let $\eta$ and $\lambda$ be the grid sizes in $u$ and $k$, respectively. Then $u_j = (j - 1) \eta$ and $k_l = (l - 1 - M/2) \lambda$ for $j, l = 1, 2, \ldots, M$, where the value of $M$ is typically a power of 2. An approximation for $C(0, T, k)$ is obtained by adopting the Trapezoid rule:

$$C(0, T, k) \approx \frac{e^{-ak}}{\pi} \sum_{j=1}^M e^{-iu_j k} \xi(0, T, u_j) \eta. \quad (1.3.5)$$

It is easy to see that the effective upper limit for the integration is

$$a = M \eta.$$
Then Equation (1.3.5) is written as

\[ C(0, T, k_l) \approx \frac{e^{-\alpha k_l}}{\pi} \sum_{j=1}^{M} e^{-iu_j k_l} \xi(0, T, u_j) \eta \quad \text{for} \quad l = 1, \ldots, M. \]  

(1.3.6)

Noting that \( u_j = (j - 1)\eta \), we write

\[ C(0, T, k_l) \approx \frac{e^{-\alpha k_l}}{\pi} \sum_{j=1}^{M} e^{-i\lambda \eta (j-1)(l-1)} e^{iM \lambda \eta / 2} \xi(0, T, u_j) \eta. \]

If

\[ \lambda \eta = \frac{2\pi}{M}, \]

then using Simpson’s rule weightings, the call price can be rewritten as

\[ C(0, T, k_l) \approx \frac{e^{-\alpha k_l}}{\pi} \sum_{j=1}^{M} e^{-i(2\pi/M)(j-1)(l-1)} e^{i(j-1)\pi} \xi(0, T, u_j) \frac{\eta}{3} (3 + (-1)^j - \delta_{j-1}), \]

where \( \delta_n \) is the Kronecker delta function.

### 1.4 Papers to be included

In this thesis, we consider the valuation of a series of options under different regime-switching models, which consist of five self-contained papers.

In Chapter 2, we investigate the valuation of two types of foreign equity options under a Markovian regime-switching mean-reversion lognormal model. The growth of globalization provides investors with opportunities to enjoy the profits of foreign investments. Such desires of investors have caused the introduction of different kinds of options on foreign assets, including the foreign equity options. Our model can be regarded as an extension of the mean-reversion lognormal model for the foreign exchange rate. In our model, we adopt a continuous-time, finite-state, observable Markov chain to represent the underlying economy and assume model parameters will change when
the states of the chain switch. A prerequisite condition of the fast Fourier transform (FFT) approach is the formulae for the characteristic function of the logarithmic underlying entity price. For the foreign equity option with a strike price in the domestic currency, the product of the foreign equity price and the foreign exchange rate is regarded as the underlying entity. Consequently, the characteristic function of the underlying entity can be derived directly by the summation of the characteristic functions of both the logarithmic foreign equity price and the logarithmic foreign exchange rate under the risk-neutral probability measure. However, the foreign equity option with a strike price in the foreign currency, a measure change technique is adopted first to take the expectation of the foreign exchange rate as the numéraire. Then the conditional characteristic function of the logarithmic equity price can be calculated via the Bayes’ rule. To illustrate the pricing of foreign equity options, we provide a numerical analysis using the FFT method. Finally, an empirical application is provided, revealing the advantages of the regime-switching model over the model without regime-switching in terms of lower fitting and prediction errors. Chapter 2 is based on the paper by Fan et al. (2014).

Chapter 3 presents discussions on the valuation of power options under a regime-switching jump-diffusion model. Comparatively little attention has been paid to the valuation of power options, one kind of exotic options. The nonlinear payoff functions will lead to higher potential profits and more suitable hedging approaches for nonlinear risks. We assume that the dynamic of the underlying asset is governed by a jump-diffusion model with regime-switching. A version of regime-switching Esscher transform is applied to change the real world measure to the risk-neutral probability measure. The fast Fourier transform is applied to calculate the prices of the power options. In the numerical analysis, we choose two specified parametric forms of the compensator measure to illustrate the implementation of our model. Chapter 3 is based on the paper
Chapter 4 discusses the option valuation under a stochastic interest rate model in the presence of regime switches. The stochastic interest rate models with regime-switching may be a practical approach to increase the long-term effectiveness of stochastic interest rate models. Consequently, we consider the option valuation under a Markovian regime-switching Hull-White model. We start with a risk-neutral probability measure. Then a measure change technique is applied to derive a forward measure with the zero-coupon bond value as the numéraire and the formulae for the zero-coupon bond. The option price can be written as a product of the zero-coupon bond value and the expectation value of the payoff under the forward measure according to the Bayes’ rule. The Gaussian feature of the integral of the stochastic interest rate helps us calculate the characteristic function of the logarithmic return. After deriving the formulae for the characteristic function of the logarithmic return, the FFT can be adopted to discretize the pricing formulae and illustrate the implementation of our approach. Chapter 4 is based on the paper by Fan et al. (2012b).

In Chapter 5, we consider the valuation of equity-linked annuities under the double regime-switching model, where model parameters are modulated by a Markov chain and a jump component of the price level of the underlying investment fund will be triggered when the states of the Markov chain switch. Earlier works on valuation of equity-linked products under regime-switching models either ignore the regime-switching risk or exogenously determine the regime-switching risk, while our model can determine the regime-switching risk endogenously. To specify a unique pricing kernel in this incomplete market setup, two measure change techniques, namely, a generalized version of regime-switching Esscher transform and the minimal martingale measure method, are adopted. Girsanov’s theorem gives us the dynamic of the underlying investment fund and the transition intensity of the Markov chain under the risk-neutral probability
measure. It is easy to see that the payoff of equity-linked products can be considered as a combination of the payoff functions of several European-style options. Consequently, the fast Fourier transform can be applied to derive the pricing formulae for the point-to-point equity-indexed annuities and variable annuities with guaranteed minimum death benefit (GMDB). The analytical pricing formulae for equity-indexed annuities and variable annuities with GMDB are practical for both academic researchers and industry practitioners. In our numerical examples, we present the prices of the point-to-point EIA and variable annuities with GMDB under both the single regime-switching model and the double regime-switching model. We also report the values of the regime-switching risk. The numerical results and the sensitivity analysis provide us with intuitive understandings of the valuation of equity-linked annuities. Chapter 5 is based on the paper by Fan et al. (2013).

In Chapter 6, we discuss the valuation of dynamic fund protection plans under regime-switching models with unobservable states of the Markov chain. To protect investors from downside risk, a dynamic fund protection plan is an effective and convenient tool. To select a pricing kernel, we employ the regime-switching Esscher transform. Then the approach of partial differential equations is applied to price the floating strike lookback put options embedded in the investment plans. Since the states of the economy cannot be observed, we have to estimate the model parameters from real data. Here, we propose a three-stage method to estimate the proposed model parameters, which is intuitively appealing and easy to implement in practice. It uses the Baum-Welch algorithm to estimate the model parameters of the hidden Markov chain, the Viterbi algorithm to select the most-probable path for the chain and the maximum likelihood techniques to estimate the model parameters. Chapter 6 is based on the paper by Fan et al. (2012c).
Chapter 2

Pricing foreign equity options with regime-switching

2.1 Introduction

Due to recent technological advance and trade liberalization, the growth of globalization has been accelerated and the economic growth has been boosted unprecedentedly. In the global financial markets, foreign exchange risk arising from fluctuations in foreign exchange rate has received much attention, especially since the currency crises in emerging markets. To hedge and manage foreign exchange risk, both academic researchers and industry practitioners have proposed a variety of currency options. Partly attributed to globalization, many firms and households are massively involved in investment activities of foreign assets. There are two key sources of risk arising in investment on foreign assets, namely foreign exchange (FX) risk and asset’s price risk. Effective management of these two sources of risk is the key to successes in foreign assets investments. Foreign equity options provide a possible way to manage or hedge
both the FX risk and the equity price risk. According to the definition in Kwok and Wong (2000), “the currency-translated foreign equity options are contingent claims whose payoffs are determined by financial prices or indices denominated in one currency but the actual payouts are settled in another currency”. As its name implies, the underlying asset of a foreign equity option is a foreign equity. There are two main tempting features of foreign equity options. Firstly, foreign equity options provide investors with a variety of flexible ways to deal with the multidimensional risks, mainly the foreign equity price fluctuation risk and the foreign exchange risk. There exist a variety types of foreign equity options with different payoff functions. Seen from this aspect, foreign equity options could provide investors with more investment and risk management choices. Exchange-trade is the second advantage of foreign equity options, which means this kind of financial product enjoys a higher degree of liquidity. Furthermore, the regulations of clearinghouse help investors reduce or avoid some risks, such as counterparty risk.

Since the pricing model of foreign equity options needs to depict the joint dynamics of the exchange rate and foreign equity prices, there are some literature about the valuation of foreign equity options under different models. Early works usually consider the valuation of foreign equity options in the Black-Scholes framework. Kwok and Wong (2000) investigated the valuation of foreign equity options with path dependent features. Examples of pricing foreign equity options beyond the traditional BS framework include a multi-dimensional Lévy process to depict the dynamics of both the exchange rate and the foreign equity prices in Huang and Hung (2005). Xu et al. (2011a) considered the valuation of foreign equity option under a stochastic volatility model with double jumps. To incorporate the impacts of skewness and kurtosis on foreign equity option prices, the Gram-Charlier series expansion approach was adopted by Xu et al. (2011b).
It is known that certain vital features of financial time series cannot be depicted by the classical Black-Scholes models. Among the models extending the classical Black-Scholes model, the ability to incorporate structural changes in economic conditions makes regime-switching models one of the most practically useful models in financial econometrics. These changes, which may be attributed to changes in economic fundamentals or business cycles, represent an additional source of risk to which an additional amount of risk premium may be required to compensate. Furthermore, the risk brought by these changes can be hardly diversified since it is more likely to be regarded as a systematic risk. Since regime-switching models provide a natural and convenient choice to model the structural changes in economic conditions, especially due to financial crises, this class of models will enjoy more and more popularity. The seminal work of Hamilton (1989) popularized applications of regime-switching models in financial econometrics. Typically, the so-called “modulated by a Markov chain” means the model dynamics or parameters will change when the underlying Markov chain changes from one state to another. The states of the Markov chain represent the states of an economy. Since the last decade or so, there has been an interest on studying option valuation problems in regime-switching models (see Naik (1993), Guo (2001), Buffington and Elliott (2002), Elliott et al. (2005), Siu (2008), Yuen and Yang (2010a), Shen et al. (2013), Shen and Siu (2013a), etc.). Considering the increasingly changing foreign exchange market, there is a considerable interest to investigate the valuation of currency options under regime-switching models, including Bollen et al. (2000), Siu et al. (2008b), Bo et al. (2010), etc. Empirical studies in Bollen et al. (2000) verified that trading strategies under regime-switching models can gain higher profit and be more attractive to investors. That also indicates the potential practical value of regime-switching models.

However, relatively little attention has been given to pricing foreign equity options
in the context of regime-switching models. In this paper, we investigate the valuation of foreign equity options under a Markovian regime-switching mean-reversion lognormal model, which extends the mean-reversion lognormal model for foreign exchange rate. More specifically, the model parameters, including the risk-free domestic interest rate, the volatility of the foreign equity, the mean-reversion level and the volatility of the foreign exchange rate, as well as the instantaneous correlation coefficient between the foreign equity and the exchange rate, are modulated by a continuous-time, finite-state, observable Markov chain. To apply the fast Fourier transform (FFT) approach to discretize the integral pricing formula, we need to first calculate the characteristic function of the logarithmic underlying equity price. For the valuation of the foreign equity option with strike price in the foreign currency \((FEO_F)\), we first apply a measure change technique and use a version of the Bayes’ rule to derive the conditional characteristic function of the logarithmic equity price under the new measure. For the valuation of the foreign equity option with strike price in the domestic currency \((FEO_D)\), we calculate the characteristic function of the summation of the logarithmic foreign equity price and the logarithmic foreign exchange rate under the risk-neutral probability measure. Then, we derive the Fourier transform of the foreign equity option price in these two cases. To illustrate the pricing of foreign equity options, we provide a numerical analysis using the FFT method. Finally, an empirical application is provided, revealing that the regime-switching model outperforms the model with a single regime in terms of lower fitting errors and prediction errors. The main contributions of this paper are as follows. (1) We investigate the valuation of foreign equity options under a regime-switching mean-reversion lognormal model. The main feature of our model is

\footnote{Following the notation in Xu et al. (2011b), let \(FEO_F\) and \(FEO_D\) represent the foreign equity option with strike price in the foreign currency and the foreign equity option with strike price in the domestic currency, respectively.}
that it combines the advantages of both regime-switching models and mean-reversion lognormal models. The mean reversion feature of foreign exchange rates has been well-documented. (Jorion and Sweeney (1996); Sweeney (2006); Wong and Lau (2008); Wong and Lo (2009); Wong and Zhao (2010); Leung et al. (2013)). (2) By applying a measure change technique, the Fourier transform of the $FEO_F$ option price can be calculated more easily. Then, we adopt the FFT approach in Carr and Madan (1999) and Liu et al. (2006) to derive closed-form pricing formulae for the foreign equity options.

The rest of the paper are organized as follows. The next section presents the model dynamics. In Section 2.3, we derive the pricing formulae of $FEO_F$ and $FEO_D$ under the Markovian, regime-switching, mean-reversion lognormal model, respectively. Section 2.4 presents numerical examples. An empirical application of our model is provided in Section 2.5. The final section concludes the paper.

2.2 The model dynamics

In this section, we consider a continuous-time economy with a finite time horizon $T := [0, T]$, where $T < \infty$. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space. In the literature about foreign exchange rate modeling, it is customary to assume that $\mathcal{P}$ is a risk-neutral probability measure (See Wong and Lau (2008)) \footnote{Due to the additional uncertainty brought by the Markov chain, the financial market under regime-switching models is generally incomplete. Consequently, there exist more than one equivalent martingale measure in the incomplete market. Researchers have proposed many approaches to determine a unique equivalent martingale measure under some criterions. For instance, we could apply a generalized version of regime-switching Esscher transform to determine an equivalent martingale measure. After changing the real world probability measure into a risk-neutral probability measure, we can derive the dynamics of both the foreign equity price and the foreign exchange rate. Then,}. To describe the
evolution of the state of an economy over time, we consider a continuous-time, $N$-state, observable Markov Chain $X := \{X(t)\} \in \mathcal{T}$. The $N$ different states of the chain may represent $N$ observable different states of an economy or different stages of a business cycle. Without loss of generality, using the convention in Elliott et al. (1994), we assume the chain $X$ has a canonical state space $E := \{e_1, e_2, \ldots, e_N\} \subset \mathbb{R}^N$, where the $j$-th component of $e_i$ is the Kronecker delta $\delta_{ij}$, for each $i, j = 1, 2, \ldots, N$. Let $Q := [q_{ij}]_{i,j=1,2,\ldots,N}$ denote the generator or rate matrix of the chain $X$ under $\mathcal{P}$, where $q_{ij}$ is the transition intensity of the chain $X$ from state $e_j$ to state $e_i$. Then the following semimartingale dynamics for the chain $X$ were obtained in Elliott et al. (1994):

$$X(t) = X(0) + \int_0^t QX(s)ds + M(t), \quad t \in \mathcal{T}.$$ 

Here $\{M(t)\} \in \mathcal{T}$ is an $\mathbb{R}^N$-valued, $(\mathbb{F}^X, \mathcal{P})$-martingale, where $\mathbb{F}^X$ is the right-continuous, $\mathcal{P}$-complete, natural filtration generated by the chain $X$.

We now specify the Markovian regime-switching models for the dynamics of the foreign equity and the foreign exchange rate. Let $S := \{S(t)\} \in \mathcal{T}$ and $Z := \{Z(t)\} \in \mathcal{T}$ denote the price process of the foreign equity and the logarithmic foreign exchange rate process respectively. Let $y'$ be the transpose of a vector or a matrix $y$, $\langle \cdot, \cdot \rangle$ be the scalar product in $\mathbb{R}^N$, and $\text{diag}(y)$ be the diagonal matrix with diagonal elements being given by the components of the vector $y$. For each $t \in \mathcal{T}$, let $r(t)$ and $\sigma(t)$ be the domestic, instantaneous continuously compounded, interest rate and the volatility of the equity at time $t$, respectively. We assume that $r(t)$ and $\sigma(t)$ are determined by the value $X(t)$ of the chain at time $t$ as:

$$r(t) := \langle r, X(t) \rangle,$$

similar calculation process can be conducted to derive the analytical formulae for the foreign equity options.
\[ \sigma(t) := \langle \sigma, X(t) \rangle , \]

where \( r := (r_1, r_2, \ldots, r_N)' \in \mathbb{R}^N \) with \( r_i > 0 \) and \( \sigma := (\sigma_1, \sigma_2, \ldots, \sigma_N)' \in \mathbb{R}^N \) with \( \sigma_i > 0 \) for each \( i = 1, 2, \ldots, N \).

Let \( \{\alpha(t)|t \in T\} \) and \( \{\gamma(t)|t \in T\} \) be the mean-reversion level and volatility of the process \( Z \). Again we suppose that

\[ \alpha(t) := \langle \alpha, X(t) \rangle , \]
\[ \gamma(t) := \langle \gamma, X(t) \rangle , \]

where \( \alpha := (\alpha_1, \alpha_2, \ldots, \alpha_N)' \in \mathbb{R}^N \) and \( \gamma := (\gamma_1, \gamma_2, \ldots, \gamma_N)' \in \mathbb{R}^N \) with \( \gamma_i > 0 \) for each \( i = 1, 2, \ldots, N \). The parameter \( \beta \), controlling the speed of mean reversion for the logarithmic foreign exchange rate process, is assumed to be a positive constant.

Then, under the risk-neutral probability \( P \), the dynamics of \( S \) and \( Z \) are given by

\[ dS(t) = r(t)S(t)dt + \sigma(t)S(t)dW_1(t) , \] (2.2.1)

and

\[ dZ(t) = \beta(\alpha(t) - Z(t))dt + \gamma(t)dW_2(t) , \] (2.2.2)

where \( \{W_1(t)|t \in T\} \) and \( \{W_2(t)|t \in T\} \) are two standard Brownian motions with respect to their respective right-continuous, \( P \)-complete, natural filtrations under \( P \). Furthermore, we suppose that the two Brownian motions \( W_1, W_2 \) are correlated and that the instantaneous correlation coefficient at time \( t \) is given by:

\[ \langle W_1, W_2 \rangle (t) = \int_0^t \rho(s)ds , \]

where \( \rho(t) = \langle \rho, X(t) \rangle \) and \( \rho := (\rho_1, \rho_2, \ldots, \rho_N)' \in \mathbb{R}^N \) with \(-1 < \rho_j < 1\) for \( j = 1, \ldots, N \); with a slight abuse of the notation, \( \{(W_1, W_2)(t)|t \in T\} \) denotes the predictable quadratic covariation process between \( W_1 \) and \( W_2 \).
For each $t \in \mathcal{T}$, write $Y(t) := \ln(S(t))$ and $F(t) := e^{Z(t)}$ for the logarithmic foreign equity price and the foreign exchange rate at time $t$, respectively. Note that in Shen and Siu (2013a), a Markovian regime-switching Hull-White model was used for modeling stochastic interest rate, so there is a positive probability that the interest rate goes negative. Here the logarithmic foreign exchange rate is modelled by a Markovian regime-switching, mean-reverting process, so that foreign exchange rate stays positive. Applying Itô’s differentiation rule, the risk-neutral dynamics of $Y$ and $F$ are given by

$$dY(t) = \left(r(t) - \frac{1}{2}\sigma^2(t)\right)dt + \sigma(t)dW_1(t),$$

and

$$dF(t) = \left(\beta\alpha(t) + \frac{1}{2}\gamma^2(t) - \beta\ln F(t)\right)dt + \gamma(t)dW_2(t).$$

Let $\mathbb{F}^S := \{\mathcal{F}^S(t) | t \in \mathcal{T}\}$, $\mathbb{F}^Z := \{\mathcal{F}^Z(t) | t \in \mathcal{T}\}$ and $\mathbb{F}^X := \{\mathcal{F}^X(t) | t \in \mathcal{T}\}$ be the right-continuous, $\mathcal{P}$-complete, natural filtrations generated by processes $S$, $Z$ and $X$, respectively. Furthermore, we define the enlarged filtration $\mathbb{G} := \{\mathcal{G}(t) | t \in \mathcal{T}\}$ by the minimal $\sigma$-field containing $\mathcal{F}^S(t)$, $\mathcal{F}^Z(t)$ and $\mathcal{F}^X(t)$. That is,

$$\mathcal{G}(t) := \mathcal{F}^S(t) \vee \mathcal{F}^Z(t) \vee \mathcal{F}^X(t), \quad t \in \mathcal{T}.$$ monde

For each $t \in \mathcal{T}$, $\mathcal{G}(t)$ represents publicly available market information up to time $t$.

**Remark 2.2.1.** The economic motivation of using the regime-switching model may be illustrated by an utility maximization problem of a representative agent’s intertemporal consumption. If the agent receives a stream of Markov-modulated dividends from holding a single security, it can be shown that the maximization of an expected utility of the agent leads to a regime-switching model dynamics for the security. Interested readers may refer to Di Graziano and Rogers (2009) for details. Due to various economic and
financial factors, the stream of dividends may change with the current economic conditions or market modes. The regime-switching model provides a natural way to describe these changes, so it may provide a better matching of the agent’s expectation towards market trends than its non-regime switching counterpart.

2.3 Pricing foreign equity options

Typically, there are two types of foreign equity options, which are classified by the strike prices of the underlying foreign equities. The payoff functions of the two kinds of foreign equity options are given by

\[ FEO_F = F(T)(S(T) - K_F)_+ , \]
and

\[ FEO_D = (F(T)S(T) - K_D)_+ , \]

where \( K_F \) and \( K_D \) are the strike prices in the foreign currency and in domestic currency, respectively. Note that the payoff is represented in the domestic currency for both the \( FEO_F \) option and the \( FEO_D \) option.

2.3.1 Valuation of an \( FEO_F \) option

Considering the particular payoff function of an \( FEO_F \) option, the following pricing formula is standard:

\[ C_F(0, T, K_F) = E[e^{-\int_0^T r(t) dt} F(T)(S(T) - K_F)_+] , \]  

where \( E \) is an expectation under the risk-neutral measure \( \mathcal{P} \). Let \( k_F = \ln(K_F) \) be the logarithmic strike price. The modified \( FEO_F \) option price is defined by

\[ c_F(0, T, k_F) = e^{\alpha_F k_F} C_F(0, T, K_F) , \]
where $a_F$ is a predetermined positive constant such that $c_F(0, T, k_F)$ is square integrable in $k_F$ over the entire real line. As in Carr and Madan (1999), the Fourier transform of $c_F(0, T, k_F)$ is as follows:

$$
\psi_F(0, T, u) = \int_{-\infty}^{\infty} e^{iuk_F} c_F(0, T, k_F) dk_F .
$$

(2.3.2)

The following proposition gives an integral representation for the price of the $FEO_F$ option.

**Proposition 2.3.1.** For each $j = 1, 2, \ldots, N$, let

$$
g_j(t, u) := -r_j + \beta e^{-\beta(T-t)} \alpha_j + \frac{1}{2} e^{-2\beta(T-t)} \sigma_j^2 - \frac{1}{2}(u - i(a_F + 1))^2 \sigma_j^2
$$

$$
+ i(u - i(a_F + 1)) \left( r_j - \frac{1}{2} \sigma_j^2 + e^{-\beta(T-t)} \rho_j \sigma_j \gamma_j \right).
$$

Write

$$
g(t, u) := (g_1(t, u), g_2(t, u), \ldots, g_N(t, u))' \in \mathcal{C}^N ,
$$

where $\mathcal{C}$ is the complex space and $\mathcal{C}^N$ is the $N$-fold product of $\mathcal{C}$.

Then under the Markovian regime-switching mean-reversion lognormal model, the price of the $FEO_F$ option is given by the following integral formula:

$$
C_F(0, T, K_F) = \frac{e^{-a_F k_F}}{\pi} \int_0^\infty e^{-iuk_F} \psi_F(0, T, u) du ,
$$

where

$$
\psi_F(0, T, u) = \frac{\exp \left\{ e^{-\beta T} Z(0) + (i u + a_F + 1) Y(0) \right\}}{a_F^2 + a_F - u^2 + i(2a_F + 1) u}
$$

$$
\left\langle X(0) \exp \left\{ \int_0^T \text{diag}(g(t, u)) dt + QT \right\}, 1 \right\rangle .
$$
Before proving Proposition 2.3.1, several useful results are given. Firstly, we introduce a probability measure $Q$ equivalent to $P$ on $G(T)$ by the following Radon-Nikodým derivative:

$$\frac{dQ}{dP}_{G(T)} := \frac{e^{Z(T)}}{E[e^{Z(T)}|\mathcal{F}^X(T)]}.$$ 

The idea of introducing a probability measure $Q$ here may not be unlike that of introducing a forward measure in a Markovian regime-switching, mean-reverting process, (see, for example, Shen and Siu (2013a)). Then, by a version of the Bayes’ rule, Eq. (2.3.1) becomes

$$C_{F}(0, T, K_{F}) = E\left[E\left[e^{-\int_{0}^{T}r(t)dt}F(T)(S(T) - K_{F})_{+}|\mathcal{F}^X(T)\right]\right]$$

$$= E\left[e^{-\int_{0}^{T}r(t)dt}E[F(T)|\mathcal{F}^X(T)]E^Q[(S(T) - K_{F})_{+}|\mathcal{F}^X(T)]\right],$$

where $E^Q$ represents an expectation under the measure $Q$.

**Lemma 2.3.1.** The Radon-Nikodým derivative is given by:

$$\frac{dQ}{dP}_{G(T)} = \exp \left\{ -\frac{1}{2} \int_{0}^{T} e^{-2\beta(T-t)}\gamma^2(t)dt + \int_{0}^{T} e^{-\beta(T-t)}\gamma(t)dW_2(t) \right\}.$$ 

Then,

$$W_1^Q(t) := W_1(t) - \int_{0}^{t} \rho(s)e^{-\beta(t-s)}\gamma(s)ds$$

and

$$W_2^Q(t) := W_2(t) - \int_{0}^{t} e^{-\beta(t-s)}\gamma(s)ds$$

are two standard Brownian motions under $Q$. The instantaneous correlation coefficient between $W_1^Q$ and $W_2^Q$ at time $t$ is still $\rho(t)$.
Proof. A direct calculation to Eq. (2.2.2) gives

\[ Z(T) = e^{-\beta T} Z(0) + \int_0^T \beta e^{-\beta (T-t)} \alpha(t) dt + \int_0^T e^{-\beta (T-t)} \gamma(t) dW_2(t) . \]

It is easy to see that given \( \mathcal{F}^X(T) \), the conditional distribution of \( Z(T) \) is a normal distribution with the following mean and variance:

\[
E[Z(T)|\mathcal{F}^X(T)] = e^{-\beta T} Z(0) + \int_0^T \beta e^{-\beta (T-t)} \alpha(t) dt ,
\]

\[
\text{Var}[Z(T)|\mathcal{F}^X(T)] = \int_0^T e^{-2\beta (T-t)} \gamma^2(t) dt ,
\]

so that

\[
E\left[ e^{Z(T)} | \mathcal{F}^X(T) \right] = \exp \left\{ e^{-\beta T} Z(0) + \int_0^T \beta e^{-\beta (T-t)} \alpha(t) dt + \frac{1}{2} \int_0^T e^{-2\beta (T-t)} \gamma^2(t) dt \right\} .
\]

Consequently,

\[
\frac{dQ}{dP} = \frac{e^{Z(T)}}{E\left[ e^{Z(T)} | \mathcal{F}^X(T) \right]} = \exp \left\{ - \frac{1}{2} \int_0^T e^{-2\beta (T-t)} \gamma^2(t) dt + \int_0^T e^{-\beta (T-t)} \gamma(t) dW_2(t) \right\} .
\]

By Girsanov’s theorem, \( W_1^Q(t) := W_1(t) - \int_0^t \rho(s) e^{-\beta (t-s)} \gamma(s) ds \) and \( W_2^Q(t) := W_2(t) - \int_0^t e^{-\beta (t-s)} \gamma(s) ds \) are two standard Brownian motions under \( Q \). It is also obvious that

\[
\langle W_1^Q, W_2^Q \rangle(t) = \langle W_1, W_2 \rangle(t) = \int_0^t \rho(u) du .
\]

To apply the fast Fourier transform approach (Carr and Madan (1999)), we need to calculate the characteristic function of the logarithmic terminal spot price of the foreign equity. Here, due to the measure change, the conditional characteristic function of the logarithmic terminal foreign equity price under the probability measure \( Q \) should be derived first.
Lemma 2.3.2. The conditional characteristic function of $Y(T)$ given $\mathcal{F}_x(T)$ under $Q$ is calculated as:

$$
\phi_{Y(T)|\mathcal{F}_x(T)}^Q(u) := E^Q[e^{iuY(T)}|\mathcal{F}_x(T)]
$$

$$
= \exp \left\{ iuY(0) + \frac{1}{2} \int_0^T \left( r(t) - \frac{1}{2} \sigma^2(t) \right) dt - \frac{1}{2} u^2 \int_0^T \sigma^2(t) dt 
+ iu \int_0^T e^{-\beta(T-t)} \rho(t) \sigma(t) \gamma(t) dt \right\},
$$

where $E^Q$ denotes an expectation under the probability measure $Q$.

Proof. It is easy to find that under the probability measure $Q$,

$$
S(T) = S(0) \exp \left\{ \int_0^T \left( r(t) - \frac{1}{2} \sigma^2(t) \right) dt + \int_0^T e^{-\beta(T-t)} \rho(t) \sigma(t) \gamma(t) dt + \int_0^T \sigma(t) dW_1^Q(t) \right\}.
$$

Consequently, conditional on $\mathcal{F}_x(T)$, $Y(T)$ is normally distributed with the following mean and variance:

$$
E^Q[Y(T)|\mathcal{F}_x(T)] = Y(0) + \int_0^T \left( r(t) - \frac{1}{2} \sigma^2(t) \right) dt + \int_0^T e^{-\beta(T-t)} \rho(t) \sigma(t) \gamma(t) dt,
$$

and

$$
\text{Var}^Q[Y(T)|\mathcal{F}_x(T)] = \int_0^T \sigma^2(t) dt.
$$

Then the conditional characteristic function of $Y(T)$ is easy to compute.

For notational simplicity, write:

$$
R_T := \int_0^T r(t) dt,
$$

$$
L_T := \int_0^T \beta e^{-\beta(T-t)} \alpha(t) dt + \frac{1}{2} \int_0^T e^{-2\beta(T-t)} \gamma^2(t) dt.
$$

The result presented in the following lemma resembles to those in Lemma 4.1 and Lemma 4.2 in Shen and Siu (2013a).

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Lemma 2.3.3. Let $F^Q_{Y(T)|\mathcal{F}^X(T)}(y)$ be the conditional distribution function of $Y(T)$ given $\mathcal{F}^X(T)$ under $Q$. Then, the Fourier transform of the price of the FEOF option is given by:

$$
\psi_F(0, T, u) = \frac{E\left[ e^{-R_T + LT + e^{-\beta T}Z(0)} \phi_Y^{Q_{Y(T)|\mathcal{F}^X(T)}}(u - i(a_F + 1)) \right] - a_F^2 + a_F - u^2 + i(2a_F + 1)u}{a_F^2 + a_F - u^2 + i(2a_F + 1)u} \exp \left\{ -\beta T Z(0) + (iu + a_F + 1)Y(0) \right\} \left< X(0) \exp \left\{ \int_0^T \text{diag}(g(t, u)) dt + QT \right\}, 1 \right>.
$$

where $\phi_Y^{Q_{Y(T)|\mathcal{F}^X(T)}}(u)$ denotes the conditional characteristic function of $Y(T)$ given $\mathcal{F}^X(T)$ under the probability measure $Q$.

Proof. Let $k_F = \ln(K_F),

$$
\psi_F(0, T, u) = \int_{-\infty}^{\infty} e^{iuk_F} c(0, T, k_F) dk_F
$$

$$
= \int_{-\infty}^{\infty} e^{a_F k_F} e^{iuk_F} C(0, T, K_F) dk_F
$$

$$
= \int_{-\infty}^{\infty} e^{a_F k_F} e^{iuk_F} E\left[ e^{-R_T} F(T)(e^{Y(T)} - e^{k_F})_+ \right] dk_F
$$

$$
= E \left\{ \int_{-\infty}^{\infty} e^{a_F k_F} e^{iuk_F} e^{-R_T} E[\mathcal{F}^X(T)] E^Q[(e^{Y(T)} - e^{k_F})_+] \right\} dk_F
$$

$$
= E \left\{ \int_{-\infty}^{\infty} e^{a_F k_F} e^{iuk_F} e^{-R_T - LT + e^{-\beta T}Z(0)} E^Q[(e^{Y(T)} - e^{k_F})_+] \right\} dk_F
$$

$$
= E \left\{ \int_{-\infty}^{\infty} e^{a_F k_F} e^{iuk_F} e^{-R_T - LT + e^{-\beta T}Z(0)} \int_{k_F}^{\infty} (e^y - e^{k_F}) F_{Y(T)|\mathcal{F}^X(T)}(dy) dk_F \right\}
$$

$$
= E \left\{ \int_{-\infty}^{\infty} (e^{-R_T - LT + e^{-\beta T}Z(0)} \int_{-\infty}^{a_F + iu} e^{(1+a_F+iu)y} dk_F F_{Y(T)|\mathcal{F}^X(T)}(dy) \right\}
$$

$$
= E \left\{ \int_{-\infty}^{\infty} (e^{-R_T - LT + e^{-\beta T}Z(0)} \int_{-\infty}^{a_F + iu} \frac{e^{(1+a_F+iu)y}}{1+a_F+iu} dk_F F_{Y(T)|\mathcal{F}^X(T)}(dy) \right\}
$$

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\[
E[e^{-R_T + L_T + e^{-\beta T} Z(0)} e^{Q_T} x_T(u - i(a_F + 1))]
\]
\[
= \frac{e^{\beta T Z(0) + (iu + a_F + 1) Y(0)}}{a_F^2 + a_F - u^2 + i(2a_F + 1)u}
\]
\[
\exp \left\{ \int_0^T \langle g(t, u), X(t) \rangle \, dt \right\}
\]

Define
\[
\Gamma(t) := X(t) \exp \left\{ \int_0^t \langle g(s, u), X(s) \rangle \, ds \right\}, \quad t \in T.
\]

Applying Itô’s differentiation rule to \( \Gamma(t) \),
\[
d\Gamma(t) = \langle g(t, u), X(t) \rangle \Gamma(t) dt + \exp \left\{ \int_0^t \langle g(s, u), X(s) \rangle \, ds \right\} dX(t)
\]
\[
= (\text{diag}(g(t, u)) + Q) \Gamma(t) dt + \exp \left\{ \int_0^t \langle g(s, u), X(s) \rangle \, ds \right\} dM(t)
\]

Taking expectation on both sides of Eq. (2.3.3) under \( P \) gives:
\[
dE[\Gamma(t)] = (\text{diag}(g(t, u)) + Q) E[\Gamma(t)] dt.
\]

Solving gives
\[
E \left[ X(T) \exp \left\{ \int_0^T \langle g(t, u), X(t) \rangle \, dt \right\} \right] = X(0) \exp \left\{ \int_0^T \text{diag}(g(t, u)) dt + QT \right\}.
\]

Consequently,
\[
E \left[ \exp \left\{ \int_0^T \langle g(t, u), X(t) \rangle \, dt \right\} \right] = \langle E[\Gamma(T)], 1 \rangle
\]
\[
= \left\langle X(0) \exp \left\{ \int_0^T \text{diag}(g(t, u)) dt + QT \right\}, 1 \right\rangle.
\]

Proof of Proposition 2.3.1. The proof is standard. Applying the inverse Fourier transform to Eq. (2.3.2), the following equation can be derived:
\[
C_F(0, T, K_F) = e^{-a_F k_F} C_F(0, T, k_F) = \frac{e^{-a_F k_F}}{\pi} \int_0^\infty e^{-iuk_F} \psi_F(0, T, u) \, du.
\]

The result can be obtained from Lemma 2.3.3 immediately. \qed
2.3.2 Valuation of an \( FEO_D \) option

The pricing formula for an \( FEO_D \) option is given by

\[
C_D(0, T, K_D) = E[e^{-\int_0^T r(t)dt} (F(T)S(T) - K_D)_+] ,
\]

(2.3.4)

where \( E \) denotes an expectation under the risk-neutral measure \( \mathcal{P} \). Let \( k_D = \ln(K_D) \).

The modified \( FEO_D \) option price is defined by

\[
c_D(0, T, k_D) = e^{a_D k_D} C_D(0, T, k_D) ,
\]

where \( a_D \) is a predetermined positive constant such that \( c_D(0, T, k_D) \) is square integrable in \( k_D \) over the entire real line. As in Carr and Madan (1999), the Fourier transform of \( c_D(0, T, k_D) \) is as follows:

\[
\psi_D(0, T, v) = \int_{-\infty}^{\infty} e^{ivk_D} c_D(0, T, k_D) dk_D .
\]

(2.3.5)

Define a process \( \{ G(t) | t \in \mathcal{T} \} \) with \( G(t) := \ln(F(t)S(t)) \) for each \( t \in \mathcal{T} \). By direct calculation,

\[
G(T) = e^{-\beta T} Z(0) + \ln S(0) + \int_0^T \left( r(t) - \frac{1}{2} \sigma^2(t) + \beta e^{-\beta(T-t)} \alpha(t) \right) dt
\]

\[+ \int_0^T \sigma(t) dW_1(t) + \int_0^T e^{-\beta(T-t)} \gamma(t) dW_2(t) .
\]

Then, the conditional characteristic function of \( G(T) \) given \( \mathcal{F}^X(T) \) under the risk-neutral probability measure \( \mathcal{P} \) is given by

\[
\phi_{G(T)|\mathcal{F}^X(T)}(v) = E[\exp\{ivG(T)\}|\mathcal{F}^X(T)] .
\]

The following proposition gives an integral representation for the price of the \( FEO_D \) option. This result resembles to that of Proposition 2.3.1.
Proposition 2.3.2. For each \( j = 1, 2, \ldots, N \), let
\[
h_j(t, v) := -r_j + i(v - i(a_D + 1)) \left( r_j - \frac{1}{2} \sigma_j^2 + \beta e^{-\beta(T-t)} \alpha_j \right)
- \frac{1}{2} (v - i(a_D + 1))^2 (\sigma_j^2 + e^{-2\beta(T-t)} \gamma_j^2 + 2e^{-\beta(T-t)} \rho_j \sigma_j \gamma_j).
\]
Write
\[
h(t, v) := (h_1(t, v), h_2(t, v), \ldots, h_N(t, v))' \in \mathcal{C}^N.
\]

Then under the Markovian regime-switching mean-reversion lognormal model, the price of the \( FEO_D \) option is given by
\[
C_D(0, T, K_D) = e^{-a_D k_D} \frac{\pi}{\psi_D(0, T, v)} \int_0^\infty e^{-ivk_D} \psi_D(0, T, v) dv,
\]
where
\[
\psi_D(0, T, v) = \exp \left\{ \frac{(iv + a_D)(Y(0) + e^{-\beta T} Z(0))}{a_D^2 + a_D - v^2 + i(2a_D + 1)v} \right\} \left\langle X(0) \exp \left\{ \int_0^T \text{diag}(h(t, v)) dt + QT \right\}, 1 \right\}.
\]

Proof. The proof here resembles to that in Lemma 2.3.3. Let \( F_{G(T)\|\mathcal{F}^X(T)}(g) \) denote the conditional distribution function of \( G(T) \) given \( \mathcal{F}^X(T) \) under \( \mathcal{P} \). Then, the Fourier transform of the \( FEO_D \) option price is given by:
\[
\psi_D(0, T, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{a_D k_D} e^{ivk_D} E[e^{-R_T (e^{G(T)} - e^{k_D})} + |\mathcal{F}^X(T)] dk_D
= E \left[ \int_{-\infty}^{\infty} e^{a_D k_D} e^{ivk_D} E[e^{-R_T (e^{G(T)} - e^{k_D})} + |\mathcal{F}^X(T)] dk_D \right]
= E \left[ \int_{-\infty}^{\infty} e^{a_D k_D} e^{ivk_D} e^{-R_T} \int_{k_D}^{\infty} (e^g - e^{k_D}) F_{G(T)\|\mathcal{F}^X(T)}(dg) dk_D \right]
= E \left[ \int_{-\infty}^{\infty} e^{-R_T} \int_{-\infty}^{g} e^{a_D k_D} e^{ivk_D} (e^g - e^{k_D}) dk_D F_{G(T)\|\mathcal{F}^X(T)}(dg) \right]
\]
The derivation of the second equation is similar with the proof in Lemma 2.3.3.

\[
\psi_D(0, T, v) = \frac{\exp\left\{ (iv + a_D + 1)(Y(0) + e^{-\beta T}Z(0)) \right\}}{a_D^2 + a_D - v^2 + i(2a_D + 1)v} E\left[ \exp\left\{ \int_0^T \langle h(t, v), X(t) \rangle \, dt \right\} \right]
\]

\[
= \frac{\exp\left\{ (iv + a_D + 1)(Y(0) + e^{-\beta T}Z(0)) \right\}}{a_D^2 + a_D - v^2 + i(2a_D + 1)v}
\left\langle X(0) \exp \left\{ \int_0^T \text{diag}(h(t, v)) \, dt + QT \right\}, 1 \right\rangle.
\]

The derivation of the second equation is similar with the proof in Lemma 2.3.3. □
2.4 Numerical examples

In this section, we perform a numerical analysis for pricing the $FEO_F$ option and the $FEO_D$ option under our regime-switching mean-reversion lognormal model. To simplify our computation, we consider a two-state Markov chain $X$. For each $t \in T$, $X(t) = (1, 0)'$ and $X(t) = (0, 1)'$ are State 1 and State 2, respectively.

The rate matrix of the Markov chain $X$ under $P$ is assumed to be

$$Q = \begin{pmatrix} -q & q \\ q & -q \end{pmatrix},$$

where $q$ takes values in $[0,1]$. Intuitively, with a larger $q$, the economy will display a more volatile feature. We consider the following configurations of other parameters given in Table 2.4.1. Here, the domestic interest rate, the mean-reversion level and volatility of the exchange rate, the volatility of the foreign equity as well as the instantaneous correlation coefficient between the foreign equity price and the exchange rate take different values when the states of the economy change. Table 2.4.2 presents the prices of the $FEO_F$ option and the $FEO_D$ option with different modified strike levels under the regime-switching mean-reversion lognormal model, where we assume $S(0) = 1$, $F(0) = 1$, $T = 1$ and $q = 0.5$. The FFT method is applied to calculate the option prices (see also Carr and Madan (1999), Lee (2004), Liu et al. (2006), Wong
and Guan (2011) and Kwok et al. (2012)). The estimation of the method consists of the following steps. Firstly, we can derive a dampened option pricing formulae by introducing a dampened coefficient $\alpha$ in order to ensure the square integrability of the dampened option price. Analytical pricing formulae can be derived via the inverse Fourier transform using the relationship between the characteristic function of the logarithmic asset price and the dampened call option price. Then, an approximation for the option price can be obtained by adopting the Trapezoid rule. To discretize the pricing formulae, we also have to determine the grid sizes, the dampened coefficient, the effective upper limit for the integration and relevant parameter values. For the mathematical derivation of the process, interested readers can refer to Section 1.3 entitled “Fast Fourier transform”.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$FEO_F$ State 1</th>
<th>$FEO_F$ State 2</th>
<th>$FEO_D$ State 1</th>
<th>$FEO_D$ State 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.3</td>
<td>0.5614</td>
<td>0.4554</td>
<td>0.9849</td>
<td>0.8450</td>
</tr>
<tr>
<td>-0.2</td>
<td>0.4672</td>
<td>0.3611</td>
<td>0.9022</td>
<td>0.7713</td>
</tr>
<tr>
<td>-0.1</td>
<td>0.3772</td>
<td>0.2698</td>
<td>0.8164</td>
<td>0.6932</td>
</tr>
<tr>
<td>0</td>
<td>0.2947</td>
<td>0.1882</td>
<td>0.7282</td>
<td>0.6112</td>
</tr>
<tr>
<td>0.1</td>
<td>0.2225</td>
<td>0.1225</td>
<td>0.6388</td>
<td>0.5269</td>
</tr>
<tr>
<td>0.2</td>
<td>0.1623</td>
<td>0.0753</td>
<td>0.5500</td>
<td>0.4426</td>
</tr>
<tr>
<td>0.3</td>
<td>0.1146</td>
<td>0.0450</td>
<td>0.4637</td>
<td>0.3610</td>
</tr>
</tbody>
</table>

As shown in Table 2.4.2, for both the $FEO_F$ option and the $FEO_D$ option, the option prices in State 1 are systematically higher than those in State 2 when the strike
level is fixed. If the option valuation is viewed from the perspective of a domestic investor, State 1 is a “Bad” state while State 2 is a “Good” one. Seen from the foreign exchange rate aspect, the higher volatility of the foreign exchange rate means higher potential profits when the income, denominated in the foreign currency, translated into the domestic currency. On the other hand, seen from the foreign equity aspect, the underlying foreign equity has a higher interest rate and a lower volatility in State 2. A lower volatility means less chance of the equity price being very high or very low. In this case, the option will be less valuable. Consequently, it is reasonable that the option prices in State 1 are higher than the corresponding prices in State 2 due to the additional amount of risk premium required to compensate for a adverse economic condition. Note that the option prices converge quickly. In our illustration, we always adopt the number of discretization $M = 4096$.

In the sequel, we use the valuation of an $FEO_F$ option as an example for sensitivity analysis. The sensitivity analysis of the valuation of an $FEO_D$ option can be conducted similarly.

### 2.4.1 The impact of $q$ on option prices

Under our model, we assume $S(0) = 1$, $F(0) = 1$ and $T = 1$. To illustrate the impact of $q$ on option prices, we perform a sensitivity analysis for the option prices with respect to the rate of transition $q$. From Figs. 2.4.1 and 2.4.2, when $q$ increases, the option prices in State 1 and State 2 display different trends. In State 1, the option prices decrease while increase with $q$ in State 2. Note that the value of $q$ calibrates the probability of the chain $X$ transiting between State 1 and State 2. As explained earlier, the options are more expensive in State 1 and cheaper in State 2. Consequently, the option price will decrease when $q$ increases in State 1, while the opposite trend is displayed in State
2. When \( q = 0 \), the regime-switching effect will not exist. Under this degenerate case, option prices are the highest in State 1 and lowest in State 2.

From Figs. 2.4.1 and 2.4.2, a particular attention is given to the case \( q = 0 \), where the model dynamics for the foreign equity and the foreign exchange rate have no switching regimes. For simplicity, we denote our regime-switching mean-reversion lognormal model and the model without regime-switching as the “RS” model and the “NRS” model respectively. Figs. 2.4.3 and 2.4.4 provide us with a visual comparison between the option prices under the RS model and the NRS model, with the assumption that \( q = 0.5 \) in the RS model. As indicated in Figs. 2.4.3 and 2.4.4, the foreign equity option prices are lower (higher) under the RS model than those under the NRS model in State 1 (State 2). This is intuitively clear if State 1 and State 2 are interpreted as a “Bad” state and a “Good” one, respectively. Compared with the NRS model, the possibility of regime shifts from the current state to the opposite one in the RS model will inevitably lower option prices in a “Bad” state while higher those in a “Good” one. In other words, ignoring the regime-switching effect would result in the \( FEO_F \).
Figure 2.4.3: Option prices calculated under the RS model and the NRS model in State 1

Figure 2.4.4: Option prices calculated under the RS model and the NRS model in State 2

option being overpriced in State 1 and being underpriced in State 2.

2.4.2 The impact of $T$ and $k$ on option prices

Figure 2.4.5: Option prices corresponding to different $T$ and $k$ in State 1

Figure 2.4.6: Option prices corresponding to different $T$ and $k$ in State 2
Figs. 2.4.5 and 2.4.6 depict the price of the $FEO_F$ option versus the modified strike value $k$ and the maturity time $T$. It is easy to see that the longer the maturity time is, the higher the option price when $k$ is fixed in both the two states. On the other hand, when the maturity time remains the same, the price of the $FEO_F$ option decreases when $k$ increases.

2.4.3 The impact of initial equity price $S(0)$ and $F(0)$ on option prices

Figs. 2.4.7 and 2.4.8 illustrate the $FEO_F$ option prices versus different initial equity price $S(0)$ and different initial foreign exchange rate $F(0)$ with $k = 1$, $q = 0.5$ and $T = 1$. When the initial equity price $S(0)$ or the initial exchange rate $F(0)$ increases, the price of the $FEO_F$ option is more likely to be higher due to the possible higher payoff. Note that the increasing speed of the $FEO_F$ option price is faster against $S(0)$ than that against $F(0)$.

![Option prices against S(0) and F(0) in State 1](image1)

![Option prices against S(0) and F(0) in State 2](image2)

Figure 2.4.7: Option prices corresponding to different $S(0)$ and $k$ in State 1

Figure 2.4.8: Option prices corresponding to different $S(0)$ and $k$ in State 2
2.4.4 The impact of the correlation coefficient $\rho_1$ on option prices

Furthermore, we provide sensitivity analysis for the correlation coefficient $\rho_1$ with different $k$ in both State 1 and State 2 under the assumption that $S(0) = 1$, $T = 1$, $F(0) = 1$ and $q = 0.5$. As illustrated in Figs. 2.4.9 and 2.4.10, the foreign equity option prices will increase as $\rho_1$ does given that other parameters are fixed. This indicates that an additional amount of premium is required to compensate the correlation risk between the foreign equity price and the exchange rate.

Figure 2.4.9: Option prices corresponding to different $\rho_1$ and $k$ in State 1

Figure 2.4.10: Option prices corresponding to different $\rho_1$ and $k$ in State 2

2.5 Empirical studies

In this section, an empirical study of the regime-switching mean-reversion lognormal model is provided to illustrate the practical implementation of the model. Here, we take the Nikkei 225 index as the foreign equity and the US dollar as the domestic currency
Figure 2.5.1: Nikkei 225 index from September 2003 to September 2013

Figure 2.5.2: USD/JPY exchange rates from September 2003 to September 2013
from the perspective of an US investor. Firstly we calibrate the model parameters to the market prices of the European call options on Nikkei 225 index and the exchange rates between the US dollar (USD) and the Japanese Yen (JPY). By comparing the in-sample fitting errors and out-of-sample prediction errors, we illustrate how well the RS model might fit the market data and how the RS model might improve on the NRS model. As in the last section, we assume that the Markov chain has only two states and use the valuation of an \( FEO_F \) option as an example.

Fig. 2.5.1 and Fig. 2.5.2 describe the value of the Nikkei 225 index and the exchange rate USD/JPY from September 2003 to September 2013. From these figures, one may see that the stock index and the exchange rate exhibit both the regime-switching and mean-reversion features. Our dataset consists of European call option prices written on the Nikkei 225 index and the exchange rate of USD/JPY for seventeen consecutive trading days from 1 October 2013 to 17 October 2013, obtained from the Datastream Database of Reuters. For each trading day, the options have 9 strikes ranging from 12000 to 16000. The in-sample data include the option prices and exchange rates from 1 October 2013 to 14 October 2013 and the out-of-sample data include the rest option prices and exchange rates from 15 October 2013 to 17 October 2013.

Without loss of generality, we assume the domestic risk-free interest rate to be \( r = (0.02, 0.04)' \). In addition, we assume the rate matrix is not necessarily symmetric. To test how well the RS model and the NRS model fit the market data, we employ the method of nonlinear least squares by minimizing the sum of squared errors between the market prices and model prices. Denote the model parameters as \( \Theta := (\alpha_1, \alpha_2, \gamma_1, \gamma_2, \beta, \sigma_1, \sigma_2, \rho_1, \rho_2, q_{12}, q_{21}, p) \). Here the model prices are the weighted averages of the option prices in State 1 and State 2, with weights being given by \( p \) and
The parameter estimates of the RS model are given by
\[ \Theta = (0.650, 0.625, 0.346, 0.201, 0.902, 0.209, 0.102, -0.697, 0.101, 0.666, 0.334, 0.315) \].

Furthermore, Table 2.5.1 reports the root mean square error (RMSE) for the fitting and prediction errors of both the RS model and the NRS model based on the same in-sample and out-of-sample data. As indicated in Table 2.5.1, the RS model has lower RMSEs for both fitting and prediction errors than the NRS model. The results verify that the RS model improve quite significantly the performance of the NRS model in the description of the dynamics of the foreign equity price and the exchange rate. From

<table>
<thead>
<tr>
<th>Errors</th>
<th>RS model</th>
<th>NRS model</th>
</tr>
</thead>
<tbody>
<tr>
<td>In-sample</td>
<td>0.4356%</td>
<td>1.0286%</td>
</tr>
<tr>
<td>Out-of-sample</td>
<td>0.9529%</td>
<td>1.9859%</td>
</tr>
</tbody>
</table>

the perspective of an investor, using the RS model for the valuation of foreign equity options can provide more accurate option prices and hence reduce model risks.

## 2.6 Conclusions

We considered the valuation of foreign equity options with strike prices in both the foreign currency and the domestic foreign currency under a regime-switching mean-reversion lognormal model. The parameters were assumed to be modulated by an observable, finite-state Markov chain. The FFT approach was applied to price these two kinds of foreign equity options. Numerical examples and empirical studies were provided to illustrate the practical implementation of our methods.
Chapter 3

Valuation of power options under a Markovian regime-switching jump-diffusion model

3.1 Introduction

Power options, a kind of exotic options, may be alien to some investors. Numerous works focus on the valuation of European options with linear terminal payoff functions since the Black-Scholes formula was introduced by Black and Scholes (1973), while relatively little attention has been paid to the valuation of power options in the literature (Kreuser and Seigel (1995); Heynen and Kat (1996); Ong (1996); Tompkins (1999)). The main feature of a power option is that its payoff is raised to a power function rather than a linear one, i.e., the payoff function is determined by a power function with a certain degree. For investors, the high leverage means a high potential payoff at maturity. This feature is one of the main reasons why power options have attracted
increasing attention in the finance industry. Furthermore, the nonlinear payoff functions of power options can provide hedging strategies to many nonlinear risks, which are more common in financial markets. Power option valuation is absolutely important from the practical perspective, since traditional European options cannot satisfy an increasing demand of investors in the ever changing financial market. On the other hand, the nonlinear payoff functions are also one of the reasons leading to higher power option prices than the corresponding traditional European option prices. Sellers may face significant losses due to the high leverage of power options.

Zhang (1997) and Haug (1998) described different categories of power options, classified by the subjects which are raised to some powers. Two main types are standard power options and powered options, which raise the price of the underlying asset and the payoff to some power, respectively. Most of the early works deal with the valuation of squared power options, the most prevalent kind of power options. Generalized closed-form pricing formulas and applications of squared power options were given in Tompkins (1999). Esser (2003, 2004) considered the valuation of both power and powered options under the Black-Scholes model and the stochastic volatility (SV) model. Blenman and Clark (2005) investigated the valuation of power exchange options. The hedging of power options were investigated by Heynen and Kat (1996) and Tompkins (1999). Examples of pricing power options via the martingale method include Su et al. (2013) and Zhao and Wang (2013). Su et al. (2013) considered a Markov-modulated Geometric Brownian motion and Zhao and Wang (2013) considered a Markov-modulated Merton jump-diffusion model.

Recently, perhaps due to the global financial crisis (GFC) of 2008, there has been much interest in incorporating the impacts of structural changes in economic conditions on financial modelling. To capture the impact of macro-economic condition shifts, a number of models have been proposed. Regime-switching models are one of the most
popular and practically useful models. The history of the regime-switching models may be traced back to the early works of Quandt (1958), Goldfeld and Quandt (1973) and Tong (1978). This class of models was popularized by Hamilton (1989) in financial econometrics. In the new millennium, there has been a considerable interest in applying regime-switching models to option valuation. Due to the presence of an additional source of uncertainty introduced by regime switches, the financial market is generally incomplete. Esscher transform, an important tool in actuarial science, has been widely adopted to select a pricing kernel since the seminal work of Gerber and Shiu (1994) (see Yang (2004) for an overview). The regime-switching Esscher transform was introduced by Elliott et al. (2005) to select a pricing kernel under a continuous-time Markovian regime-switching market. In Siu et al. (2006), the price dynamics of the risky assets were assumed to be governed by a multivariate Markov chain and the pricing formula was derived. Elliott et al. (2007) investigated the option pricing under a generalized Markovian jump-diffusion model. Under the assumption of a generalized regime-switching jump diffusion model, Siu et al. (2008a) discussed the valuation of participating life insurance products. To price options under a discrete-time double Markovian regime-switching (DMRS) model, Siu et al. (2011) applied a discrete-time regime-switching Esscher transform. Examples of pricing exotic options under regime-switching models include Boyle and Draviam (2007), Ching et al. (2007). Note that Ching et al. (2007) considered the valuation of exotic options under a high-order Markovian regime switching model. Adopting a trinomial tree method, the option valuation under a jump-diffusion model with regime-switching was investigated in the literature (Yuen and Yang (2009)).

Fourier transform has wide applications of option valuation. Carr and Madan (1999) introduced the fast Fourier transform (FFT). The main feature of the FFT approach is that it computes the discrete Fourier transform (DFT) much faster than

In this paper, we consider the valuation of general power options under a Markovian regime-switching jump-diffusion model. In such model, parameters, including the appreciation rate, the volatility, the risk-free interest rate and the intensity of the jump process, will have different values when the states of an observable, continuous-time, finite-state Markov chain switch. Applying the FFT method, we develop a method to value a power option under a regime-switching jump-diffusion model. Compared with Liu et al. (2006), our contributions are as follows. Firstly, we employ the Esscher transform to select a pricing kernel and derive the martingale condition, while Liu et al. (2006) considered the option valuation directly under the risk-neutral probability measure. Furthermore, we incorporate a jump process in the pricing dynamics of the underlying risky asset. In the numerical examples, we calculate the option prices under the model with particular forms of the compensator measure and the Markovian regime-switching intensity measure, including the Markov-modulated generalized Gaussian process and the Markov-modulated Merton jump-diffusion model. For comparative purpose, the option prices under a Markovian regime-switching model without jumps are also presented. Compared with Su et al. (2013) and Zhao and Wang (2013), we considered a generalized Markovian regime-switching jump-diffusion model, which is an extension of the Markov-modulated Geometric Brownian motion in Su et al. (2013) and the Markov-modulated Merton jump-diffusion model in Zhao and Wang (2013). Furthermore, the fast Fourier transform method can be applied to a regime-switching model with multi-regimes, while the application of the martingale method in Su et
al. (2013) and Zhao and Wang (2013) are comparatively difficult to be extended to a multi-state regime-switching model.

The rest of the paper is structured as follows. The following section presents the model dynamics. In Section 3.3, we select a pricing kernel using the Esscher transform. Section 3.4 presents the valuation of power options using the FFT approach. Firstly, we derive the characteristic function of the logarithmic asset price. We then use the fast Fourier transform (FFT) method to determine the prices of options. In Section 3.5, we give a numerical example to illustrate the valuation of power options using the FFT approach. Section 3.6 concludes the paper.

3.2 The model dynamics

We consider a continuous-time financial market with two primitive assets, namely, a zero-coupon bond $B$ and a risky asset $S$. The time horizon $\mathcal{T}$ is assumed to be finite, i.e., $\mathcal{T} := [0, T]$, where $T < \infty$. Uncertainties in our modelling framework are described by a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is a real-world probability measure. We model structural changes of the states of an economy by a continuous-time, finite-state, observable Markov chain $X := \{X(t) | t \in \mathcal{T}\}$ on $(\Omega, \mathcal{F}, \mathcal{P})$. The finite state space $S := \{s_1, s_2, \ldots, s_N\}$ is interpreted as the state space of an economy. Without loss of generality, using the convention in Elliott et al. (1994), we consider the canonical state space $\mathcal{E} := \{e_1, e_2, \ldots, e_N\}$, where the $j$-th component of $e_i$ is the Kronecker delta $\delta_{ij}$, for each $i, j = 1, 2, \ldots, N$. Let $Q := [q_{ij}]_{i,j=1,2,\ldots,N}$ be the rate matrix of the chain $X$ under $\mathcal{P}$, where $q_{ij}$ is a constant transition intensity of the chain $X$ from state $e_j$ to state $e_i$. Note that $q_{ii} = -\sum_{j=0,j\neq i}^{N} q_{ij}$, $i = 0, 1, \ldots, N$. Let $\mathcal{F}^X := \{\mathcal{F}^X(t) | t \in \mathcal{T}\}$ be the right-continuous, $\mathcal{P}$-complete, natural filtration generated by the chain $X$. Elliott
et al. (1994) obtained the following semi-martingale dynamics for the chain:

\[ X(t) = X(0) + \int_0^t QX(s)ds + M(t) \], \quad t \in \mathcal{T},

where \{M(t)\,|\,t \in \mathcal{T}\} is an \( \mathbb{R}^N \)-valued \((\mathcal{F}^X, \mathcal{P})\)-martingale.

Let \( Z := \{Z(t)\,|\,t \in \mathcal{T}\} \) be a real-valued, purely random jump process on \((\Omega, \mathcal{F}, \mathcal{P})\). Then, by convention,

\[ Z(t) = Z(0) + \sum_{0 < s \leq t} (Z(s) - Z(s^-)) = \sum_{0 < s \leq t} \Delta Z(s), \quad t \in \mathcal{T}, \]

where \( \Delta Z(t) := Z(t) - Z(t^-) \), \( Z(0) = Z(0^-) = 0 \).

Following Elliott et al. (2007) and Siu et al. (2008a), we define a Poisson random measure \( \gamma(\cdot, \cdot) \) on the product space \((\mathcal{T} \times \mathbb{R}^+, \mathcal{B}(\mathcal{T}) \otimes \mathcal{B}(\mathbb{R}^+))\), where \( \mathcal{B}(\mathcal{T}) \) and \( \mathcal{B}(\mathbb{R}^+) \) are the Borel \( \sigma \)-fields generated by open subsets of \( \mathcal{T} \) and \( \mathbb{R}^+ \), respectively. The random measure selects random jump times \( u \) and random jump sizes \( z := \Delta Z(u) \) of the return from the asset price \( S \) and is given by

\[
\gamma(du, dz) = \sum_{k \geq 1} \delta(\mathcal{T}_k, \Delta Z(\mathcal{T}_k)) (du, dz) \mathbb{1}_{\{\mathcal{T}_k < \infty, \Delta Z(\mathcal{T}_k) \neq 0\}}.
\]

Here the function \( \delta \) is the random Dirac delta function; \( \mathbb{1}_A \) is the indicator function of an event \( A \). Indeed, a random measure should depends on the element \( \omega \), i.e., \( \omega \in \Omega \). To simplify the notation, we suppress \( \omega \) and write \( \gamma(dt, dz) \) instead of \( \gamma(\omega, dt, dz) \).

Let \( \kappa_j(dz) \) denote a Lévy measure on the space \( \mathbb{R}^+ \) when the Markov chain is in the \( j \)-th state, \( j = 1, 2, \ldots, N \). To ensure the existence of the kernel-biased completely random measure to be defined in the sequel (see James (2002) and James (2005)), we assume that for an arbitrary strictly positive function on \( \mathbb{R}^+ \), \( h \) and \( \kappa_j \) are selected in such a way that for each bounded set \( \mathcal{K} \) in \( \mathcal{T} \),

\[
\sum_{j=1}^N \int_{\mathcal{K}} \int_{\mathbb{R}^+} \min(h(z), 1)\kappa_j(dz) dt < \infty.
\]
We assume $\nu_{X(t)}(dt, dz)$ to be the compensator of the random measure $\gamma(dt, dz)$.

$$
\nu_{X(t)}(dt, dz) := \kappa_{X(t)}(dz)dt = \sum_{j=1}^{N} \langle X(t), e_j \rangle \kappa_j(dz)dt .
$$

Let $\tilde{\gamma}(dt, dz)$ denote the compensated version of the random measure $\gamma(dt, dz)$.

$$
\tilde{\gamma}(dt, dz) = \gamma(dt, dz) - \nu_{X(t)}(dt, dz) .
$$

The instantaneous market interest rate of the zero-coupon bond is given by:

$$
r(t) := \langle r, X(t) \rangle ,
$$

where $r := (r_1, r_2, \ldots, r_N)$ with $r_j > 0$ for each $j = 1, 2, \ldots, N$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^N$. Then the dynamics of the zero-coupon bond $\{B(t)| t \in \mathcal{T}\}$ is given by

$$
\frac{dB(t)}{B(t)} = r(t)dt, \quad B(0) = 1 .
$$

Let $\{Y(t)| t \in \mathcal{T}\}$ be the return process, i.e., $Y(t) := \ln(S(t)/S(0))$ for each $t \in \mathcal{T}$, where $S(t)$ is the price of the risky asset $S$ at time $t$. Here, we propose a Markov-modulated jump-diffusion process for modelling the dynamics of the underlying risky asset. The dynamics of the return process are given by:

$$
Y(t) = \int_0^t (\mu(s) - \frac{1}{2}\sigma^2(s))ds + \int_0^t \sigma(s)dW(s) + \int_0^t \int_{\mathbb{R}^+} h(z)\tilde{\gamma}(ds, dz) ,
$$

$Y(0) = 0, \quad \mathcal{P} - a.s. .

The appreciation rate $\mu(t)$ and the volatility $\sigma(t)$ are modulated by the chain $X$ as follows:

$$
\mu(t) := \langle \mu, X(t) \rangle , \quad \sigma(t) := \langle \sigma, X(t) \rangle , \quad t \in \mathcal{T} .
$$
Here, $\mu = (\mu_1, \mu_2, \ldots, \mu_N)' \in \mathbb{R}^N$ and $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)' \in \mathbb{R}^N$ for each $t \in \mathcal{T}$, where $\mu_j > r_j$ and $\sigma_j > 0$ for all $j = 1, 2, \ldots, N$; the process $\{W(t) | t \in \mathcal{T}\}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$. Note that $W$, $\gamma$ and $X$ are mutually independent under $\mathcal{P}$.

Then, the dynamics of the risky asset $S := \{S(t) | t \in \mathcal{T}\}$ can be written as an exponential of the jump-diffusion process $Y$:

$$S(t) = S(0) \exp(Y(t)), \quad t \in \mathcal{T}, \quad S(0) = 1.$$  

Applying Itô’s differentiation rule (see e.g. Elliott (1982)),

$$\frac{dS(t)}{S(t-)} = \mu(t)dt + \sigma(t)dW(t) + \int_{\mathbb{R}^+} (e^{h(z)} - h(z) - 1) \gamma(dt, dz) + \int_{\mathbb{R}^+} h(z) \tilde{\gamma}(dt, dz).$$

Let $\mathcal{F}^S = \{\mathcal{F}^S(t) | t \in \mathcal{T}\}$ and $\mathcal{F}^Y = \{\mathcal{F}^Y(t) | t \in \mathcal{T}\}$, be the natural filtrations generated by $\{S(t) | t \in \mathcal{T}\}$ and $\{Y(t) | t \in \mathcal{T}\}$, respectively. Also, we assume that all of the filtrations given above are right continuous and $\mathcal{P}$–complete. Note either $\mathcal{F}^Y$ or $\mathcal{F}^S$ could be used as an observed information filtration. Here, we choose $\mathcal{F}^Y$. We define the filtration $\mathcal{G} = \{\mathcal{G}(t, s) | s \leq t \in \mathcal{T}\}$ by letting the double indexed $\sigma$-field $\mathcal{G}(t, s) := \mathcal{F}^X(t) \vee \mathcal{F}^Y(s)$, for any $s, t \in \mathcal{T}$ with $s \leq t$. We write $\mathcal{G}(t) = \mathcal{G}(t, t)$, for all $t \in \mathcal{T}$, and $\mathcal{G} = \{\mathcal{G}(t) | t \in \mathcal{T}\}$.

### 3.3 Pricing via the Esscher transform

The market described in Section 3.2 is incomplete. Consequently, there exist more than one equivalent martingale measure. Following Siu et al. (2008a), we shall determine an equivalent martingale measure adopting the regime-switching Esscher transform. Let $BM(\mathcal{T})$ denote the collection of $\mathcal{B}(\mathcal{T})$-measurable and nonnegative functions with compact support on $\mathcal{T}$. Here $\mathcal{B}(\mathcal{T})$ represents the Borel $\sigma$-field of $\mathcal{T}$. For each process $\theta \in BM(\mathcal{T})$, we assume that the following conditions hold:
1. For each \( t \in \mathcal{T} \), \( \theta(t) := (\theta, X(t)), \) where \( \theta := (\theta_1, \theta_2, \ldots, \theta_N)' \in \mathbb{R}^N \);

2. \( \theta \) is integrable with respect to \( Y \).

For each \( t \in \mathcal{T} \), write

\[
(\theta \cdot Y)(t) := \int_0^t \theta(s)dY(s).
\]

Define the Laplace functional of \( Y \) with respect to \( \theta \), denoted by \( \mathcal{L}_Y(\theta)(t) \), as follows:

\[
\mathcal{L}_Y(\theta)(t) := \mathbb{E}[e^{-(\theta \cdot Y)(t)}|\mathcal{F}^X(T)],
\]

where \( \mathbb{E} \) is an expectation under \( \mathcal{P} \).

**Lemma 3.3.1.** For each \( t \in \mathcal{T} \), define a \( \mathcal{G} \)-adapted process \( \Lambda := \{\Lambda(t)|t \in \mathcal{T}\} \) as follows:

\[
\Lambda(t) := \frac{e^{-(\theta \cdot Y)(t)}}{\mathcal{L}_Y(\theta)(t)}.
\]

Then,

\[
\Lambda(t) = \exp \left\{ -\int_0^t \theta(s)\sigma(s)dW(s) - \frac{1}{2} \int_0^t \theta^2(s)\sigma^2(s)ds \\
- \int_0^t \int_{\mathbb{R}^+} \theta(s)h(z)\hat{\gamma}(ds,dz) \\
- \int_0^t \int_{\mathbb{R}^+} (e^{-\theta(s)h(z)} + \theta(s)h(z) - 1)\nu_X(s)(ds,dz) \right\}.
\]

Clearly, \( \Lambda \) is a \( (\mathcal{G}, \mathcal{P}) \)-martingale.

**Proof.** By Itô’s differentiation rule for jump-diffusion processes, under \( \mathcal{P} \),

\[
e^{-(\theta \cdot Y)(t)} = 1 - \int_0^t e^{-(\theta \cdot Y)(s)}\theta(s)\left(\mu(s) - \frac{1}{2}\sigma^2(s)\right)ds \\
+ \frac{1}{2} \int_0^t e^{-(\theta \cdot Y)(s)}\theta^2(s)\sigma^2(s)ds - \int_0^t e^{-(\theta \cdot Y)(s)}\theta(s)\sigma(s)dW(s)
\]
\[ + \int_0^t \int_{\mathbb{R}^+} e^{-(\theta Y)(s)} \left( e^{-\theta h(z)} - 1 \right) \gamma(ds, dz) \]
\[ + \int_0^t \int_{\mathbb{R}^+} e^{-(\theta Y)(s)} \left( e^{-\theta h(z)} + \theta h(z) - 1 \right) \nu_X(ds, dz) . \]

Conditioning both sides on \( \mathcal{F}_X(T) \) under \( \mathcal{P} \),

\[ \mathbb{E}[e^{-(\theta Y)(t)} | \mathcal{F}_X(T)] \]
\[ = 1 - \int_0^t \mathbb{E}[e^{-(\theta Y)(s)} | \mathcal{F}_X(T)] \theta(s) \left( \mu(s) - \frac{1}{2} \sigma^2(s) \right) ds \]
\[ + \frac{1}{2} \int_0^t \mathbb{E}[e^{-(\theta Y)(s)} | \mathcal{F}_X(T)] \theta^2(s) \sigma^2(s) ds \]
\[ + \int_0^t \int_{\mathbb{R}^+} \mathbb{E}[e^{-(\theta Y)(s)} | \mathcal{F}_X(T)] (e^{-\theta h(z)} + \theta h(z) - 1) \nu_X(ds, dz) . \]

Solving then gives

\[ \mathcal{L}_Y(\theta)(t) = \exp \left\{ - \int_0^t \left( \mu(s) - \frac{1}{2} \sigma^2(s) \right) \theta(s) ds + \frac{1}{2} \int_0^t \theta^2(s) \sigma^2(s) ds \right. \]
\[ + \int_0^t \int_{\mathbb{R}^+} \left( e^{-\theta h(z)} + \theta h(z) - 1 \right) \nu_X(ds, dz) \} . \]

Hence,

\[ \Lambda(t) = \exp \left\{ - \int_0^t \theta(s) \sigma(s) dW(s) - \frac{1}{2} \int_0^t \theta^2(s) \sigma^2(s) ds \right. \]
\[ - \int_0^t \int_{\mathbb{R}^+} \theta(s) h(z) \gamma(ds, dz) \]
\[ - \int_0^t \int_{\mathbb{R}^+} \left( e^{-\theta h(z)} + \theta h(z) - 1 \right) \nu_X(ds, dz) \} . \]

Since under \( \mathcal{P} \),

\[ \mathbb{E} \left[ \frac{\Lambda(t)}{\Lambda(s)} \big| G(s) \right] = \mathbb{E} \left[ \exp \left\{ - \int_s^t \theta(u) \sigma(u) dW(u) - \frac{1}{2} \int_s^t \theta^2(u) \sigma^2(u) du \right. \right. \]
\[ - \int_s^t \int_{\mathbb{R}^+} \left( e^{-\theta(u) h(z)} + \theta(u) h(z) - 1 \right) \nu_X(u, du, dz) \]
which indicates $\Lambda$ is a $(\mathcal{G}, \mathcal{P})$-martingale.

Then, we define a new probability measure $\mathcal{Q} \sim \mathcal{P}$ on $\mathcal{G}(T)$ by the following Radon-Nikodym derivative:

$$
\left. \frac{d\mathcal{Q}}{d\mathcal{P}} \right|_{\mathcal{G}(T)} = \Lambda(T).
$$

According to the fundamental theorem of asset pricing, “the absence of arbitrage is ‘essentially’ equivalent to the existence of an equivalent martingale measure under which discounted asset prices are (local)-martingales” (Harrison and Kreps (1979) and Harrison and Pliska (1981, 1983)). The following proposition presents a necessary and sufficient condition for the local-martingale condition.

**Proposition 3.3.1.** Define the discounted price of the underlying asset as follows:

$$
\tilde{S}(t) := \exp \left\{ - \int_0^t r(s) ds \right\} S(t), \quad t \in \mathcal{T}.
$$

The discounted price process $\tilde{S} := \{\tilde{S}(t) | t \in \mathcal{T}\}$ is a $(\mathcal{G}, \mathcal{Q})$-(local)-martingale if and only if the process $\{\theta(t) | t \in \mathcal{T}\}$ satisfies the following equation:

$$
\mu(t) - r(t) - \theta(t) \sigma^2(t) + \int_{\mathbb{R}^+} (e^{-\theta(t)h(z)}(e^{h(z)} - 1) - h(z)) \kappa_{X(t)}(dz) = 0, \quad \text{for each } t \in \mathcal{T}(3.3.1)
$$

**Proof.** The proof follows that in Siu et al. (2008a).

The following lemma gives the return process under the equivalent martingale measure $\mathcal{Q}$ specified by the Esscher transform.

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Proposition 3.3.2. Under $Q$, 
\[
dY(t) = \left( r(t) - \frac{1}{2} \sigma^2(t) \right) dt + \sigma(t) dW^\theta(t) \\
- \int_{\mathbb{R}_+} (e^{h(z)} - 1 - h(z)) \nu^\theta_X(dt, dz) + \int_{\mathbb{R}_+} h(z) \tilde{\gamma}^\theta(dt, dz),
\]
where $dW^\theta(t) := dW(t) + \theta(t) \sigma(t) dt$ is a standard Brownian motion under $Q$. Suppose $\gamma^\theta(\cdot, \cdot)$ is a random measure with the following compensator:
\[
\nu^\theta_X(dt, dz) := e^{-\theta(h(z))} \nu_X(dt, dz).
\]
Write $\tilde{\gamma}^\theta(\cdot, \cdot)$ for the compensated version of $\gamma^\theta(\cdot, \cdot)$, i.e.,
\[
\tilde{\gamma}^\theta(dt, dz) := \gamma^\theta(dt, dz) - \nu^\theta_X(dt, dz).
\]

Proof. Let $P(s) \in BM(T)$,
\[
\Lambda(t) e^{-(P \cdot Y)(t)} = \exp \left\{ M_1(t) - \int_0^t P(s) \mu(s) ds + \frac{1}{2} \int_0^t P^2(s) \sigma^2(s) ds \\
+ \int_0^t P(s) \theta(s) \sigma^2(s) ds + \frac{1}{2} \int_0^t P(s) \sigma^2(s) ds \right\} \\
\exp \left\{ M_2(t) + \int_0^t \int_{\mathbb{R}_+} [e^{-(\theta(h(z)) + P(s))h(z)} - e^{-\theta(h(z))} + \theta(h(z)) \nu_X(s, dz)] ds, dz \right\},
\]
where
\[
M_1(t) = - \int_0^t (\theta(s) + P(s)) \sigma(s) dW(s) - \frac{1}{2} \int_0^t (\theta(s) + P(s))^2 \sigma^2(s) ds,
\]
\[
M_2(t) = - \int_0^t \int_{\mathbb{R}_+} (\theta(s) + P(s)) \sigma(s) h(z) \tilde{\gamma}(ds, dz) \\
- \int_0^t \int_{\mathbb{R}_+} (e^{-\theta(h(z) + P(s))} - (\theta(s) + P(s)) h(z) - 1) \nu_X(s, dz) ds, dz.
\]
Then, applying a version of the Bayes' rule,
\[
\mathcal{L}_Y(P)(t) := E^Q[e^{-(P \cdot Y)(t)}| \mathcal{F}_X(t)]
\]
\[
\begin{align*}
E[\Lambda(t)e^{-(P-Y)(t)}|\mathcal{F}^X(t)]
&= \exp \left\{ -\int_0^t P(s) \left[ \mu(s) - \theta(s)\sigma^2(s) - \frac{1}{2}\sigma^2(s) \right] ds \\
&\quad + \int_0^t \int_{\mathbb{R}^+} (e^{-\theta(s)h(z)}(e^{-P(s)h(z)} - 1) - P(s)h(z))\nu_X(s)(ds,dz) \\
&\quad + \frac{1}{2} \int_0^t P^2(s)\sigma^2(s)ds \right\}.
\end{align*}
\]

Considering the martingale condition,
\[
\mathcal{L}_Y(P)(t) = \exp \left\{ -\int_0^t P(s) \left[ r(s) - \frac{1}{2}\sigma^2(s) \right] ds + \frac{1}{2} \int_0^t P^2(s)\sigma^2(s)ds \\
&\quad + \int_0^t \int_{\mathbb{R}^+} ((e^{h(z)} - 1) - h(z))P(s)\nu_X^\theta(ds,dz) \\
&\quad + \int_0^t \int_{\mathbb{R}^+} (e^{-P(s)h(z)} - 1 + P(s)h(z))\nu_X^\theta(ds,dz) \right\}.
\]

Then the dynamics of \( Y(t) \) can be derived.

To simplify the notation, we define
\[
\begin{align*}
dY_1(t) &= \left( r(t) - \frac{1}{2}\sigma^2(t) \right) dt + \sigma(t)dW^\theta(t), \\
dY_2(t) &= \int_{\mathbb{R}^+} h(z)\gamma^\theta(dt,dz) + \int_{\mathbb{R}^+} (1 - e^{h(z)})\nu_X^\theta(dt,dz).
\end{align*}
\]

Note that the risk-neutral probability \( \mathcal{Q} \) is selected by the Esscher transform. Then, the probability law of the chain \( X \) remains the same after the measure change, i.e., under \( \mathcal{Q} \), \( X \) still has the semi-martingale dynamics:
\[
X(t) = X(0) + \int_0^t QX(u)du + M(t), \quad t \in \mathcal{T}.
\]

Recently, Siu (2011) has showed that in a regime-switching diffusion setting, an optimal equivalent martingale measure selected by the minimal relative entropy approach does not price the regime-switching risk.
3.4 Valuation of a power option using the fast Fourier transform

In this section, we derive the price of a power option under regime-switching models. Here, we adopt the Fourier methods, proposed by Carr and Madan (1999), to value the option.

Consider a power call option written on $S$ with strike $K > 0$ and maturity $T > 0$. Under the risk-neutral probability measure $Q$, the option price $C(0, T)$ at time 0 is given by

$$C(0, T) = \mathbb{E}^Q \left[ \exp \left( - \int_0^T r(t) dt \right) (e^{mY(T)} - K)_+ \right] ,$$

where $\mathbb{E}^Q[\cdot]$ is an expectation under $Q$, $m$ is the power that the asset price is raised and $m < \infty$. When $m = 1$, the valuation of a power option is equivalent to pricing a corresponding standard European call option. Write $k = \ln(K)$, then the above equation can be written as:

$$C(k) = \mathbb{E}^Q \left[ \exp \left( - \int_0^T r(t) dt \right) (e^{mY(T)} - e^k)_+ \right] .$$

3.4.1 The Fourier transform of a power option

Following the notation in Carr and Madan (1999) and Liu et al. (2006), the dampened call price is defined by

$$c(k) := \exp(\alpha k)C(k) ,$$

where $\alpha$ is called the dampening coefficient and assumed to be positive. Then the dampened call transform is given by:

$$\xi(u, 0, T) = \int_{\mathbb{R}} \exp(iuk)c(k)dk .$$
Let $F_{Y(T)|X(T)}(y)$, $F_{Y_1(T)|X(T)}(y_1)$ and $F_{Y_2(T)|X(T)}(y_2)$ denote the conditional distribution function of $Y(T)$, $Y_1(T)$ and $Y_2(T)$ given $X(T)$ under $Q$, respectively. Then the conditional characteristic function of $Y(T)$ given $X(T)$ under $Q$ is calculated as

$$
\phi_{Y(T)|X(T)}(u) = \mathbb{E}^Q \left[ e^{iuY(T)} | X(T) \right] = \int_{\mathbb{R}} e^{iuy} F_{Y(T)|X(T)}(dy) .
$$

Similarly, we define

$$
\phi_{Y_1(T)|X(T)}(u) = \mathbb{E}^Q \left[ e^{iuY_1(T)} | X(T) \right] = \int_{\mathbb{R}} e^{iuy_1} F_{Y_1(T)|X(T)}(dy_1), \quad t \in T ,
$$

$$
\phi_{Y_2(T)|X(T)}(u) = \mathbb{E}^Q \left[ e^{iuY_2(T)} | X(T) \right] = \int_{\mathbb{R}} e^{iuy_2} F_{Y_2(T)|X(T)}(dy_2), \quad t \in T .
$$

For notation simplicity, we denote $\phi_1^t(u) := \phi_{Y_1(t)|X(T)}(u)$ and $\phi_2^t(u) := \phi_{Y_2(t)|X(T)}(u)$ for each $t \in T$.

Write

$$
R_T = \int_0^T r(u) du ,
$$

$$
V_T = \int_0^T \sigma^2(u) du .
$$

and define $T_j := \int_0^T \langle X(t), e_j \rangle dt$ for $j = 1, 2, \ldots, N$. The processes can be written as:

$$
R_T = \sum_{j=1}^{N-1} (r_j - r_N)T_j + r_NT , \quad (3.4.1)
$$

$$
V_T = \sum_{j=1}^{N-1} (\sigma^2_j - \sigma^2_N)T_j + \sigma^2_NT , \quad (3.4.2)
$$

Then the Fourier transform of $c(k)$ is given by

$$
\xi(u, 0, T) = \int_{\mathbb{R}} e^{iuk} c(k) dk
$$
\[
\int_{\mathbb{R}} e^{iuk} e^{\alpha k} E^Q [e^{-RT} (e^{mY(T)} - e^{k})_{+}] dk
\]

\[
= E^Q \left[ \int_{\mathbb{R}} e^{iuk} e^{\alpha k} E^Q [e^{-RT} (e^{mY(T)} - e^{k})_{+}] \mathcal{F}(T) dk \right]
\]

\[
= E^Q \left[ \int_{\mathbb{R}} e^{-RT} e^{iuk} \int_{k}^{\infty} (e^{my} - e^{k}) F_{Y(T)|\mathcal{F}(T)} (dy) dk \right]
\]

\[
= E^Q \left[ \int_{\mathbb{R}} e^{-RT} F_{Y(T)|\mathcal{F}(T)} (dy) \int_{-\infty}^{y} (e^{my} e^{(\alpha + iu)k} - e^{(1+\alpha + iu)k}) dk \right]
\]

\[
= E^Q \left[ \frac{e^{-RT} F_{Y(T)|\mathcal{F}(T)} (u - i(\alpha + m))}{\alpha + iu} - \frac{E^Q [e^{-RT} \phi_{Y(T)|\mathcal{F}(T)} (u - i(\alpha + 1))]}{1 + \alpha + iu} \right] \tag{3.4.3}
\]

When \( m = 1 \), Equation (3.4.3) reduces to the formula of the corresponding European call option.

To apply the Fourier transform, we will first define the characteristic function of the logarithmic price. The following lemma presents the conditional characteristic functions of \( Y_1(T) \) and \( Y_2(T) \) given \( \mathcal{F}(T) \) under \( Q \):

**Lemma 3.4.1.** 1. The conditional characteristic function of \( Y_1(T) \) given \( \mathcal{F}(T) \) under \( Q \) is:

\[
\phi_{1T}(u) = \exp \left( iu \left( R_T - \frac{1}{2} V_T - \frac{1}{2} u^2 V_T \right) \right). \tag{3.4.4}
\]

2. The conditional characteristic function of \( Y_2(T) \) given \( \mathcal{F}(T) \) under \( Q \) is:

\[
\phi_{2T}(u) = \exp \left\{ \sum_{j=1}^{N} \int_{0}^{T} \langle X(s), e_j \rangle \psi_j(u) ds \right\} = \exp \left\{ \sum_{j=1}^{N} \psi_j(u) T_j \right\},
\]

where

\[
\psi_j(u) = \int_{\mathbb{R}^+} [(e^{iu h(z)} - 1) - iu (e^{h(z)} - 1)] e^{-\theta_j h(z)} \kappa_j(z) dz.
\]
Proof. 1. Clearly, given $F^X(T)$, $Y_1(T)$ follows a Gaussian distribution with mean $(R_T - \frac{1}{2} V_T)$ and variance $V_T$ under $Q$. Then the conditional characteristic function of $Y_1(T)$ given $F^X(T)$ under $Q$ is as follows:

$$\phi^1_T(u) = \exp \left( iu \left( R_T - \frac{1}{2} V_T \right) - \frac{1}{2} u^2 V_T \right).$$

2. Applying Itô’s differentiation rule to $e^{iuY_2(t)}$,

$$e^{iuY_2(t)} = e^{iuY_2(0)} + \int_0^t \int_{\mathbb{R}^+} e^{iuY_2(s^-)} (e^{iuh(z)} - 1) \gamma^\theta(ds, dz)$$

$$-iu \sum_{j=1}^N \int_0^t \int_{\mathbb{R}^+} \langle X(s), e_j \rangle e^{iuY_2(s^-)} e^{-\theta_j h(z)} (e^{h(z)} - 1) \kappa_j(z) dz ds .$$

Then the conditional characteristic function of $Y_2(T)$ given $F^X(T)$ under $Q$ is as follows:

$$\phi^2_T(u) = \mathbb{E}_Q[e^{iuY_2(T)} | F^X(T)]$$

$$= 1 + \sum_{j=1}^N \int_0^T \int_{\mathbb{R}^+} \langle X(s), e_j \rangle \phi^2_s(u)$$

$$\left[ (e^{ih(z)} - 1) - iu(e^{h(z)} - 1) \right] e^{-\theta_j h(z)} \kappa_j(z) dz ds$$

$$= 1 + \sum_{j=1}^N \int_0^T \langle X(s), e_j \rangle \phi^2_s(u) \psi_j(u) ds ,$$

where

$$\psi_j(u) = \int_{\mathbb{R}^+} \left[ (e^{ih(z)} - 1) - iu(e^{h(z)} - 1) \right] e^{-\theta_j h(z)} \kappa_j(z) dz .$$

Consequently,

$$\phi^2_T(u) = \exp \left\{ \sum_{j=1}^N \int_0^T \langle X(s), e_j \rangle \psi_j(u) ds \right\} = \exp \left\{ \sum_{j=1}^N \psi_j(u) T_j \right\} .$$

\qed
Denote \( \phi^l_T(u) := E^Q[e^{iuY_l(T)}|\mathcal{F}^X(T)] \) for \( l = 1, 2 \). According to Girsanov’s theorem, it is clear that processes \( Y_1 \) and \( Y_2 \) are conditional independent given \( \mathcal{F}^X(T) \) under \( Q \). Then, the following equation is satisfied:

\[
\phi_{Y(T)|\mathcal{F}^X(T)}(u) = \phi^1_T(u) \cdot \phi^2_T(u) .
\]

From Lemma 3.4.1, we can calculate \( \phi^l_T(u) \) for \( l = 1, 2 \).

The following two lemmas present the closed-form expressions for \( e^{-Ru} \phi^1_T(u - i(1 + \alpha)) \) and \( \phi^2_T(u - i(1 + \alpha)) \).

**Lemma 3.4.2.** For \( \alpha > 0 \) and \( u \in \mathbb{R} \),

1.

\[
e^{-Ru} \phi^1_T(u - i(1 + \alpha)) = \exp(B_1(u)T) \exp \left( i \sum_{j=1}^{N-1} A_1(u,j)T_j \right), \tag{3.4.5}
\]

where, for each \( j = 1, 2, \ldots, N - 1 \),

\[
A_1(u,j) = \left[ (r_j - r_N) + \left( \frac{1}{2} + \alpha \right) \left( \sigma^2_j - \sigma^2_N \right) \right] u + \frac{1}{2} u^2 (\sigma^2_j - \sigma^2_N)i \\
- \alpha (r_j - r_N) + \frac{1}{2} \alpha (1 + \alpha) (\sigma^2_j - \sigma^2_N) i ,
\]

\[
B_1(u) = iu \left( r_N + \left( \frac{1}{2} + \alpha \right) \sigma^2_N \right) - \frac{1}{2} u^2 \sigma^2_N \\
+ \alpha r_N + \frac{1}{2} \alpha (\alpha + 1) \sigma^2_N .
\]

2.

\[
\phi^2_T(u - i(1 + \alpha)) = \exp(B_2(u)T) \exp \left( i \sum_{j=1}^{N-1} A_2(u,j)T_j \right) ,
\]

where, for each \( j = 1, 2, \ldots, N - 1 \),

\[
A_2(u,j) = -i[\psi_j(u - i(1 + \alpha)) - \psi_N(u - i(1 + \alpha))],
\]

\[
B_2(u) = \psi_N(u - i(1 + \alpha)) .
\]
Proof. 1. From Eqs. (3.4.1), (3.4.2) and (3.4.4), it is easy to see that

\[ \phi_1^T(u - i(1 + \alpha)) \]
\[ = \exp \left( i(u - i(1 + \alpha)) \left( R_T - \frac{1}{2} V_T \right) - \frac{1}{2} (u - i(1 + \alpha))^2 V_T \right) \]
\[ = \exp \left( iu \left( R_T + \left( \alpha + \frac{1}{2} \right) V_T \right) - \frac{1}{2} u^2 V_T + (1 + \alpha) R_T + \frac{1}{2} \alpha(1 + \alpha) V_T \right) \]
\[ = \exp \left( iu \left( \sum_{j=1}^{N-1} (r_j - r_N) T_j + r_N T \right) + iu \left( \alpha + \frac{1}{2} \right) \left( \sum_{j=1}^{N-1} (\sigma_j^2 - \sigma_N^2) T_j + \sigma_N^2 T \right) \right. \]
\[ - \frac{1}{2} u^2 \left( \sum_{j=1}^{N-1} (\sigma_j^2 - \sigma_N^2) T_j + \sigma_N^2 T \right) + (1 + \alpha) \left( \sum_{j=1}^{N-1} (r_j - r_N) T_j + r_N T \right) \]
\[ \left. + \frac{1}{2} \alpha(1 + \alpha) \left( \sum_{j=1}^{N-1} (\sigma_j^2 - \sigma_N^2) T_j + \sigma_N^2 T \right) \right) . \]

Then,

\[ e^{-Rr \phi_1^T(u - i(1 + \alpha))} = \exp(B_1(u)T) \exp \left( i \sum_{j=1}^{N-1} A_1(u,j) T_j \right) , \]

where

\[ A_1(u,j) = \left[ (r_j - r_N) + \left( \frac{1}{2} + \alpha \right) (\sigma_j^2 - \sigma_N^2) \right] u + \frac{1}{2} u^2 (\sigma_j^2 - \sigma_N^2) i \]
\[ - \left[ \alpha(r_j - r_N) + \frac{1}{2} \alpha(1 + \alpha)(\sigma_j^2 - \sigma_N^2) \right] i , \]
\[ B_1(u) = iu \left( r_N + \left( \frac{1}{2} + \alpha \right) \sigma_N^2 \right) - \frac{1}{2} u^2 \sigma_N^2 \]
\[ + ar_N + \frac{1}{2} \alpha(\alpha + 1) \sigma_N^2 . \]

2. From the conditional characteristic function of \( Y_2(T) \) given \( \mathcal{F}_X(T) \) under \( Q \), we can derive the following equation:

\[ \phi_2^T(u - i(1 + \alpha)) \]
\[
\begin{aligned}
&= \exp \left\{ \sum_{j=1}^{N} \psi_j (u - i(1 + \alpha)) T_j \right\} \\
&= \exp \left\{ \sum_{j=1}^{N-1} (\psi_j (u - i(1 + \alpha)) - \psi_{N}(u - i(1 + \alpha))) T_j + \psi_{N}(u - i(1 + \alpha)) T \right\} \\
&= \exp(B_2(u)T) \exp \left( i \sum_{j=1}^{N-1} A_2(u,j) T_j \right),
\end{aligned}
\]

where, for each \( j = 1, 2, \ldots, N - 1, \)

\[
A_2(u,j) = -i[\psi_j (u - i(1 + \alpha)) - \psi_{N}(u - i(1 + \alpha))],
\]

\[
B_2(u) = \psi_{N}(u - i(1 + \alpha)).
\]

\[\square\]

**Lemma 3.4.3.** For each \( u \in \mathbb{R}, \alpha > 0 \) and \( m \in (0, \infty), \)

1. \[
e^{-Rt} \phi_1^1 (u - i(m + \alpha)) = \exp(B_3(u)T) \exp \left( i \sum_{j=1}^{N-1} A_3(u,j) T_j \right),
\]

where, for each \( j = 1, 2, \ldots, N - 1, \)

\[
A_3(u,j) = \left( r_j - r_N \right) + \left( m + \alpha - \frac{1}{2} \right) (\sigma_j^2 - \sigma_N^2) u + \frac{1}{2} u^2 (\sigma_j^2 - \sigma_N^2) i
\]

\[
- \left[ (m + \alpha - 1)(r_j - r_N) + \frac{1}{2} (m + \alpha)(m + \alpha - 1)(\sigma_j^2 - \sigma_N^2) \right] i,
\]

\[
B_3(u) = iu \left( \sigma_N^2 + \left( m + \alpha - \frac{1}{2} \right) \sigma_N^2 \right) - \frac{1}{2} u^2 \sigma_N^2
\]

\[
+ (m + \alpha - 1)r_N + \frac{1}{2} (m + \alpha)(m + \alpha - 1) \sigma_N^2.
\]

2. \[
\phi_2^2 (u - i(m + \alpha)) = \exp(B_4(u)T) \exp \left( i \sum_{j=1}^{N-1} A_4(u,j) T_j \right),
\]

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where, for each $j = 1, 2, \ldots, N - 1$,

$$A_4(u, j) = -i[\psi_j(u - i(m + \alpha)) - \psi_N(u - i(m + \alpha))] ,$$

$$B_4(u) = \psi_N(u - i(m + \alpha)) .$$

Proof. The calculation process is similar with Lemma 3.4.2.

Lemma 3.4.4. For each $t \in T$, let $J(t, T) := (J_1(t, T), J_2(t, T), \ldots, J_N(t, T)) \in [t, T]^\otimes N$, where $J_i(t, T)$ is the occupation time of the chain $X$ in state $e_i$ in the interval $[t, T]$, and $\phi(t) := (\phi_1(t), \phi_2(t), \ldots, \phi_N(t)) \in \mathcal{R}^N$. Suppose $\Phi_{J(t, T)|G(t)}(\phi(t))$ is the conditional characteristic function of the vector of occupation times $J(t, T)$ given $G(t)$ under $Q$ evaluated at the vector of transform variables $\phi(t)$. That is,

$$\Phi_{J(t, T)|G(t)}(\phi(t)) := E_Q[\exp(i \langle\phi(t), J(t, T)\rangle)|G(t)] .$$

Let $\text{diag}(\phi(t))$ denote the diagonal matrix with diagonal elements given by the components of $\phi(t)$. Then

$$\Phi_{J(t, T)|G(t)}(\phi(t)) = \left\langle X(t) \exp\left\{ \int_t^T (Q + i\text{diag}(\phi(s)))ds \right\}, 1 \right\rangle ,$$

where $1 := (1, 1, \ldots, 1)' \in \mathcal{R}^N$.

Proof. The proof is a modification of that of Lemma 5.1 in Elliott and Siu (2013). Define, for each $t \in T$, an $\mathcal{R}^N$-valued process $\{W(t, u)|u \in [t, T]\}$ by setting:

$$W(t, u) := \exp(i \langle\phi(t), J(t, u)\rangle)X(u)$$

$$= \exp\left( \int_t^u i \langle\phi(s), X(s)\rangle ds \right)X(u) .$$

Consequently,

$$dW(t, u) = i \langle\phi(u), X(u)\rangle W(t, u)du + \exp(i \langle\phi(t), J(t, u)\rangle)dX(u) .$$
Note that under $Q$, 

$$dX(u) = QX(u-)du + dM(u) ,$$

and that 

$$\langle \phi(u), X(u) \rangle W(t,u) = \text{diag}(\phi(u))W(t,u) , \quad t \in T .$$

Then, 

$$dW(t,u) = (Q + i\text{diag}(\phi(u)))W(t,u)du + \exp(i \langle \phi(t), J(t,u) \rangle)dM(u) .$$

Consequently, 

$$W(t,u) = W(t,t) + \int_t^u (Q + i\text{diag}(\phi(s)))W(t,s)ds 
+ \int_t^u \exp(i \langle \phi(t), J(t,s) \rangle)dM(s) 
= X(t) + \int_t^u (Q + i\text{diag}(\phi(s)))W(t,s)ds 
+ \int_t^u \exp(i \langle \phi(t), J(t,s) \rangle)dM(s) .$$

Conditioning both sides on $G(t)$ under $Q$ gives 

$$\mathbb{E}^Q[ W(t,u) | G(t)] = X(t) + \int_t^u (Q + i\text{diag}(\phi(s)))\mathbb{E}^Q[ W(t,s) | G(t)]ds .$$

Solving gives: 

$$\mathbb{E}^Q[ W(t,u) | G(t)] = X(t) \exp \left\{ \int_t^u (Q + i\text{diag}(\phi(s)))ds \right\} .$$

Now, 

$$\Phi_{J(t,T)|G(t)}(\phi(t)) = \mathbb{E}^Q[ \exp(i \langle \phi(t), J(t,T) \rangle) | G(t)] 
= \mathbb{E}^Q[ \exp(i \langle \phi(t), J(t,T) \rangle) \langle X(T), 1 \rangle | G(t)]$$
\[ \begin{align*}
&= E^Q[\exp(i \langle \phi(t), J(t, T) \rangle) X(T), 1] \mid \mathcal{G}(t)] \\
&= \langle E^Q[\exp(i \langle \phi(t), J(t, T) \rangle) X(T)] \mid \mathcal{G}(t), 1 \rangle \\
&= \langle E^Q[W(t, T)] \mid \mathcal{G}(t), 1 \rangle \\
&= \left\langle X(t) \exp \left\{ \int_t^T (Q + i \text{diag}(\phi(s))) ds \right\}, 1 \right\rangle .
\end{align*} \]

From Lemma 3.4.2, Lemma 3.4.3 and Lemma 3.4.4, we can derive the following proposition:

**Proposition 3.4.1.** The analytical form of \( \xi(u, 0, T) \) is given by

\[
\xi(u, 0, T) = \frac{1}{\alpha + i u} \exp(B(u)T) \\
\left\langle X(t) \exp \left\{ \int_t^T (Q + i \text{diag}(A(u, 1)(s), \ldots, A(u, N - 1)(s), 0)) ds \right\}, 1 \right\rangle \\
- \frac{1}{1 + \alpha + i u} \exp(B'(u)T) \\
\left\langle X(t) \exp \left\{ \int_t^T (Q + i \text{diag}(A'(u, 1)(s), \ldots, A'(u, N - 1)(s), 0)) ds \right\}, 1 \right\rangle ,
\]

where \( A(u, j) = A_1(u, j) + A_2(u, j) \), \( B(u) = B_1(u) + B_2(u) \), \( A'(u, j) = A_3(u, j) + A_4(u, j) \) and \( B'(u) = B_3(u) + B_4(u) \).

**Proof.** From Eq. (3.4.5) and Lemma 3.4.4, we get the analytical form of the first part of \( \xi(u, 0, T) \) as follows:

\[
\frac{1}{\alpha + i u} \exp(B(u)T) E^Q \left[ \exp \left( i \sum_{j=1}^{N-1} A(u, j)T_j \right) \right] ,
\]

Since

\[
E^Q \left[ \exp \left( i \sum_{j=1}^{N-1} A(u, j)T_j \right) \mid \mathcal{G}(t) \right]
\]
\[ = \langle X(t) \exp \left\{ \int_t^T (Q + i \text{diag}(A(u,1)(s), A(u,2)(s), \ldots, A(u,N-1)(s), 0)) ds \right\}, 1 \rangle, \]
the first part of \( \xi(u,0,T) \) can be calculated. The second part of \( \xi(u,0,T) \) can be derived similarly.

3.4.2 Valuation of a power option using the FFT

In this subsection, we discuss the use of the FFT approach to price options. To compute the Fourier transform, the characteristic function of the logarithmic asset price is needed. Assuming the characteristic function of the logarithmic asset price is known analytically, the option prices can be calculated using the FFT method.

As in Carr and Madan (1999) and Liu et al. (2006), the pricing formula of the call option can be written as:

\[
c(k) = \exp\left(-\alpha k\right) \frac{1}{\pi} \int_0^\infty \exp(-iuk) \xi(u,0,T) du .
\]

Let \( u_n = \eta(n-1) \). The FFT method is an efficient algorithm for computing the following equation:

\[
\omega(k) = \sum_{n=1}^{M} e^{-i\frac{2\pi}{M}(n-1)(k-1)} x(n), \quad \text{for} \quad k = 1, \ldots, M ,
\]

where \( M \) is typically selected to be a power of 2.

Firstly, the integral part of Eq. (3.4.7) needs to be written as an application of the summation in Eq. (3.4.8). Adopting the Trapezoid rule, an approximation for \( c(k) \) is obtained as follows:

\[
c(k) \approx \frac{\exp(-\alpha k)}{\pi} \sum_{n=1}^{M} e^{-iu_n k} \xi(u_n,0,T) \eta .
\]

Write \( b = \frac{M\lambda}{2} \). Employing a regular spacing size, the FFT returns \( M \) values of the modified logarithmic strike \( k \).

\[
k_v = -b + \lambda(v-1), \quad \text{for} \quad v = 1, \ldots, M .
\]
Then, Eq. (3.4.9) is written as:

\[
c(k_v) \approx \frac{\exp(-\alpha k_v)}{\pi} \sum_{n=1}^{M} e^{-iu_n(-b+\lambda(v-1))} \xi(u_n, 0, T) \eta
\]

\[
= \frac{\exp(-\alpha k_v)}{\pi} \sum_{n=1}^{M} e^{-i\lambda(n-1)(v-1)} e^{ibu_n} \xi(u_n, 0, T) \eta, \quad \text{for } v = 1, \ldots, M.
\]

Note that to apply the FFT, the following restriction needs to be considered:

\[
\lambda \eta = \frac{2\pi}{M}.
\] (3.4.10)

**Remark 3.4.1.** The FFT approach is significantly faster than the other approaches. To apply the FFT approach, a discrete sum (3.4.9) is used to approximate (3.4.7), leading to approximation errors. To control the approximation errors, Carr and Madan (1999) discussed the selection of the upper limit of the integral. Liu et al. (2006) showed that the errors are considerably small.

**Remark 3.4.2.** Heynen and Kat (1996) and Tompkins (1999) discussed the hedging of power options, including static hedging and dynamic hedging, when the dynamic of the underlying asset price process is a geometric Brownian motion. To deal with the feature of the nonlinear payoff, the static hedging strategy could be made by using European-style options to piecewisely linear approximate the nonlinear function. Note that the approximation cannot provide a perfect hedge position. On the other hand, typical delta hedging strategies were also discussed in the existing literature. However, in our model, the additional uncertainty brought by the modulating Markov chain and the jump risk lead to an incomplete financial market. A contingent claim cannot be perfectly replicated by a self-financing strategy in an incomplete market, which means the contingent claim cannot be perfectly hedged. The hedging of power options under regime-switching models needs to consider both the nonlinear payoff function and the
incomplete market under regime-switching models. Mean-variance hedging is one of the most popular hedging methods to deal with this in incomplete market. There are two types of mean-variance hedging, namely, variance-minimizing hedging and (locally)-risk minimizing hedging. Compared with the limitations of variance-minimizing hedging, risk-minimizing hedging strategies have attracted a considerable interest since Föllmer and Sondermann (1986) first introduced the theory of risk-minimization. Until now, many literatures considered (locally) risk-minimizing hedging strategies for contingent claims under regime-switching models, including Elliott and Osakwe (2006), Deshpande and Ghosh (2008), Elliott and Siu (2010), Basak et al. (2011), Elliott et al. (2011b), Qian et al. (2011), Su et al. (2012), etc.

3.5 Numerical examples

We consider a simple situation where there are only two states of the continuous-time, finite-state Markov chain $X$, i.e., State 1 and State 2, representing a “Bad” economy and a “Good” economy, respectively. Write $X(t) = (1, 0)'$ and $X(t) = (0, 1)'$, $\forall t \in T$, for State 1 and State 2. Then, the generator of the chain $X$ under $Q$ is given by

$$Q = \begin{pmatrix} -q & q \\ q & -q \end{pmatrix},$$

where $q$ takes values in $[0,1]$. Note that the rate matrix $Q$ is symmetric, which means the probabilities of the chain transiting from State 1 (State 2) to State 2 (State 1) within a fixed period are equal. We consider the following configurations of the parameters values.

$$r = (r_1, r_2)' = (0.02, 0.04)', \quad \sigma = (\sigma_1, \sigma_2)' = (0.4, 0.2)', \quad \mu = (\mu_1, \mu_2)' = (0.04, 0.05)' .$$

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To illustrate the valuation of power options, we specify particular forms of the kernel function and the Markovian regime-switching intensity measure. The initial price of the underlying asset is assumed to be $S(0) = 1$. For the FFT methods\footnote{The estimation method can be referred to Section 1.3 entitled “Fast Fourier transform”.}, from the restriction Equation (3.4.10), the choice of $\eta$ affects the accuracy of the approach. Here, we assume $\eta = 0.2441$ in our quadrature. For the choice of the dampening coefficient in the transform of the modified call price, we use a value of $\alpha = 1.5$\footnote{To obtain a square integrable function, the dampening coefficient $\alpha$ is selected and the dampened call pricing formula is defined. The problem how to choose the value of the coefficient $\alpha$ is considered in Carr and Madan (1999).}. In the following examples, we set the kernel function $h(z) = z$ and choose particular parametric forms of the compensator measure.

1. We first focus on the Markov-modulated inverse Gaussian (MIG) process, a special case of the Markov-modulated generalized Gamma (MGG) process when the shape parameter $\alpha = 1/2$. For notation simplicity, we denote the model with this compensator measure as “MIG model”. Suppose that the scale parameter $b(t) := b(t, X(t))$ is modulated by the Markov chain $X$ as follows:

$$b(t) := \langle b, X(t) \rangle,$$

where $b := (b_1, b_2) \in \mathbb{R}^2$ and $b_j \geq 0$, for each $j = 1, 2$. Then, as in Siu et al. (2008), the Markov-switching Lévy measure is given by

$$\kappa_{X(t)}(dz) = \frac{1}{\Gamma(1 - \alpha)} e^{-b(t)z} z^{-\alpha - 1} dz.$$

When $\alpha = 1/2$, the MGG becomes a MIG process. In this case, the martingale condition is given by

$$\mu(t) - r(t) - \theta(t)\sigma^2(t) + \int_{\mathbb{R}^+} (e^{-\theta(t)z}(e^z - 1) - z) \left( \frac{1}{\Gamma(1/2)} e^{-(b, X(t)) z - \frac{3}{2}} \right) dz = 0,$$
then under $\mathcal{Q}$, the dynamic of $Y$ is

$$dY(t) = \left( r(t) - \frac{1}{2}\sigma^2(t) \right) dt + \sigma(t)dW^\theta(t)$$

$$+ \int_{\mathbb{R}^+} \left( 1 - e^z \right) \left( \frac{1}{\Gamma(1/2)} e^{-\left( \theta + (\theta + (b, X(t))) \right) z} z^{-\frac{3}{2}} \right) dz dt + \int_{\mathbb{R}^+} z^\gamma(dt, dz),$$

where $\gamma^\theta(dt, dz)$ is a Poisson random measure with Markov-switching compensator,

$$\left( \frac{1}{\Gamma(1/2)} e^{-\left( \theta + (b, X(t))) \right) z} z^{-\frac{3}{2}} \right) dz dt.$$

Here, we suppose that the scale parameter $b$ takes the following values:

$$b = (b_1, b_2)' = (50, 20)'.$$

Furthermore, we assume $q = 0.5$, $M = 4096$ and $T = 1$ and illustrate the changes of power option prices when the strike value $K$ and the power $m$ change (see Figs. 3.5.1 and 3.5.2). Note the computation of the power option prices only takes several seconds, indicating that FFT has a quick convergence rate.
2. Under the real probability measure $\mathcal{P}$, we consider the following Markov-modulated Merton jump-diffusion model:

$$dY(t) = \left(\mu(t) - \frac{1}{2}\sigma^2(t)\right)dt + \sigma(t)dW(t) - \lambda(t)E[e^{Z_1} - 1]dt + \sum_{i=1}^{N(t)} Z_i.$$ 

Here, we assume the jump process is a Poisson process $N := \{N(t)|t \in T\}$ with Markov-modulated intensity $\lambda := \{\lambda(t)|t \in T\}$, i.e., $\lambda(t) = \langle \lambda, X(t) \rangle$. The jump amplitude $Z(t)$ is normally distributed with mean $a$ and standard deviation $c$, which are also modulated by $X$. For simplicity, we denote the model as “LN model” and assume the jump amplitude $Z(t)$ is normally distributed with the following mean $a$ and standard deviation $c$:

$$a = (a_1, a_2)' = (0.5, 0.5)', \quad c = (c_1, c_2)' = (0.2, 0.1)', \quad \lambda = (\lambda_1, \lambda_2)' = (2, 1)'.$$ 

Figs. 3.5.3 and 3.5.4 illustrate the valuation of power options when $q = 0.5$, $M = 4096$ and $T = 1$.

Figure 3.5.3: Option prices with different power $m$ and strike value $K$ in State 1 under the LN model

Figure 3.5.4: Option prices with different power $m$ and strike value $K$ in State 2 under the LN model
From Figs. 3.5.1, 3.5.2, 3.5.3 and 3.5.4, the price of a power option decreases with $K$ and increases with $m$ in both State 1 and State 2 when $q$ and $T$ remain the same. It is worth mentioning that the speed of increase in the price of the power option corresponding to an increasing $m$ is rather fast, which depicts the nonlinear feature of the power option payoff.

Table 3.5.1 illustrates the power call option prices across a set of strikes under the MIG model, the LN model and the no-jump model, respectively. Here, we assume $q = 0.5$, $S(0) = 1$, $T = 1$, $m = 2$ and $M = 4096$. Our paper is concerned with the power options under the Markovian regime-switching jump-diffusion model. For comparative purpose, we also present the option prices under the Markovian regime-switching model without jumps, which is considered in Liu et al. (2006). As indicated in Table 3.5.1, for the same strike, the price of a power option in State 1 is higher than that in State 2 under the three models. This makes intuitive sense. The main feature of State 1 is a lower interest rate and a higher volatility, while the interest rate is higher and volatility is lower in State 2. It is not unreasonable that the price of the power option in State 1 (a “Bad” economy) is higher, since additional risk premiums are required to compensate for a “Bad” economy. From the other aspect, the power option price is systematically higher under the model with jumps than that under the model without jumps. This also makes intuitive sense due to the additional amount of risk premium required to compensate for the jump risk.

From Figs. 3.5.5, 3.5.6, 3.5.7 and 3.5.8, assuming $T = 1$, $m = 2$ and $M = 4096$, we notice that the price of a power call option decreases with $q$ in State 1 while increases with $q$ in State 2. When $q$ increases, the probability of the chain $X$ transiting from State 1 to State 2 will increase. As explained earlier, the power options are cheaper in State 2. Thus, the price of the power option at State 1 (State 2) decreases (increases) as the probability of the chain transiting from State 1 to State 2 increases. This is the
Table 3.5.1: Prices calculated by FFT when $q = 0.5$ and $m = 2$

<table>
<thead>
<tr>
<th>Strikes</th>
<th>MIG model</th>
<th>LN model</th>
<th>No-jump model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$</td>
<td>State 1</td>
<td>State 2</td>
<td>State 1</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6284</td>
<td>0.5395</td>
<td>0.7446</td>
</tr>
<tr>
<td>0.7</td>
<td>0.5658</td>
<td>0.4587</td>
<td>0.6887</td>
</tr>
<tr>
<td>0.8</td>
<td>0.5135</td>
<td>0.3917</td>
<td>0.6423</td>
</tr>
<tr>
<td>0.9</td>
<td>0.4670</td>
<td>0.3368</td>
<td>0.6005</td>
</tr>
<tr>
<td>1</td>
<td>0.4225</td>
<td>0.2871</td>
<td>0.5601</td>
</tr>
</tbody>
</table>

Figure 3.5.5: Option prices corresponding to different $q$ under the MIG model with $K = 0.6, 0.7$

Figure 3.5.6: Option prices corresponding to different $q$ under the MIG model with $K = 0.8, 0.9$

reason why the power options are more expensive when $q$ decreases (increases) in State 1 (State 2). Note that the probability of the chain transiting from State 1 (State 2) to State 2 (State 1) is zero when $q = 0$. Under this special case, the regime-switching effect does not exist. In State 1, the price of a power option is maximum when $q = 0$, while the minimum price is derived when $q = 0$ in State 2.
3.6 Conclusions

In this paper, the FFT method was utilized to price power options under regime-switching jump-diffusion models, where the model parameters were modulated by a continuous-time finite-state, observable Markov chain. We first selected a risk-neutral probability measure using the regime-switching Esscher transform. Then under the risk-neutral probability measure, we calculated the characteristic function of the logarithm asset price. Once the analytical form of the characteristic function was obtained, the FFT was used to compute the option prices. Both a theoretical analysis and numerical examples illustrate the valuation process.
Chapter 4

An FFT approach for option pricing under a regime-switching stochastic interest rate model

4.1 Introduction

Option valuation has long been an important problem in the theory and practice of financial economics. A major breakthrough in this area was made by three giants in finance, Fischer Black, Myron Scholes and Robert Merton, (see Black and Scholes (1973) and Merton (1973)). Despite of the practical importance of the Black-Scholes-Merton model, its underlying assumptions, including the constant interest rate and volatility, are not justified empirically. Furthermore, interest rates have been more and more volatile in the past few decades. Many stochastic interest rate models have been introduced. Some popular models include those proposed by Vasicek (1977), Cox et al. (1985), Hull and White (1990), and others. The main feature of these models is that
the short rate process, commonly described as a diffusion or jump-diffusion process, is mean-reverting. This means that the short rate process will eventually revert to a long term value. This property is a “stylized” fact of the empirical behavior of interest rates.

However, it is worth noting that the effectiveness of the previous mentioned stochastic interest rate models is not long-lasting. Structural changes in economic conditions should be considered when modelling the short rate process. Regime-switching models can incorporate the impacts of changes in economic conditions on financial and economic dynamics. This class of models was popularized by Hamilton (1989) in financial econometrics. Since then, there has been a considerable interest in applying regime-switching models to option valuation. One direction which maybe of a practical interest is to study option valuation under a stochastic interest rate environment with regime-switching. Elliott and Mamon (2003) considered a Vasicek model with the mean reverting level being modulated by a continuous-time finite-state Markov chain, while a regime-switching Hull-White model was considered in Elliott and Wilson (2007). Using the concept of stochastic flows, Elliott and Siu (2009) discussed a bond valuation problem under a regime-switching Hull-White short rate model and a regime-switching Cox-Ingersoll-Ross model. Elliott et al. (2011) developed regime-switching term-structure models and exponential-affine forms of bond prices. Note that the short rate models investigated in the aforementioned works are diffusion-type processes. Siu (2010) proposed a general short rate model incorporating jumps of the interest rate due to some extraordinary market events or economic cycles. More specifically, Siu (2010) derived a bond pricing formula under a jump-augmented Vasick model, a kind of jump-diffusion-type short rate models, using techniques in stochastic flows. Shen and Siu (2013a) employed a partial differential equations approach to derive an exponential affine formula of a zero-coupon bond.
In this paper, we discuss the valuation of European options under a stochastic interest rate model in the presence of regime switches. We suppose that the stochastic evolution of an interest rate process over time is governed by a Markovian regime-switching Hull-White model. Under this model, the mean-reversion level, the volatilities of both an underlying risky asset price and an interest rate as well as the correlation coefficient, are modulated by an observable, continuous-time finite-state Markov chain. An approach based on the fast Fourier Transform (FFT) is used to numerically evaluate a price of a European-style call option. The main advantage of the FFT approach is that it computes the discrete Fourier transform (DFT) much faster than other approaches. Application of the FFT approach to derivative valuation has attracted much attention in both the theory and practice of quantitative finance. Černý (2004) discussed applications of the FFT in finance. Carr and Madan (1999) considered the variance gamma (VG) model and illustrated the FFT algorithm for pricing European options under the VG model. Examples of option valuation by the FFT algorithm include Benhamou (2002) and Dempster and Hong (2002). Using the FFT algorithm, Liu et al. (2006) investigated the valuation of options in a regime-switching model.

The rest of the paper is organized as follows. The next section presents the model dynamics. In Section 4.3, we derive a bond pricing formula and the related forward measure under the Markovian regime-switching Hull-White model. Section 4.4 discusses the valuation of options using the FFT approach. Firstly, we derive the characteristic function of the logarithmic terminal asset price. We then use the fast Fourier transform (FFT), introduced in Carr and Madan (1999), to compute the price of an option. In Section 4.5, we give a numerical example to illustrate the valuation of European-style options using the FFT approach. Section 4.6 concludes the paper.
4.2 The model dynamics

In this section, we consider a continuous-time economy with a finite time horizon $T$, i.e., $T := [0, T]$, where $T < \infty$. Suppose $(\Omega, \mathcal{F}, \mathcal{P})$ is a complete probability space, where $\mathcal{P}$ is a risk-neutral probability measure \footnote{Here, we start with a risk-neutral probability as in some literature on stochastic interest rate models.}. We assume the state of an economy is modelled by a continuous-time, finite-state, observable Markov Chain $X := \{X(t) | t \in T\}$. The state space of the chain is denoted by $S := \{s_1, s_2, \ldots, s_N\}$, representing $N$ different observable states of an economy. Without loss of generality, using the convention in Elliott et al. (1994), we identify the state space of the chain with a finite set of standard unit vectors $E := \{e_1, e_2, \ldots, e_N\} \subset \mathbb{R}^N$, where the $j$-th component of $e_i$ is the Kronecker delta $\delta_{ij}$, for each $i, j = 1, 2, \ldots, N$. Let $Q := [q_{ij}]_{i,j=1,2,\ldots,N}$ denote the generator or rate matrix of the chain $X$. Then Elliott et al. (1994) obtained the following semi-martingale dynamics for the chain $X$:

$$X(t) = X(0) + \int_0^t QX(s)ds + M(t), \quad t \in T.$$ 

Here $\{M(t) | t \in T\}$ is an $\mathbb{R}^N$-valued martingale with respect to the filtration generated by $X$ under the measure $\mathcal{P}$.

We now introduce the Markovian regime-switching models for the dynamics of the underlying asset value and the stochastic interest rate. Let $y'$ be the transpose of a vector or a matrix $y$. Denote $\{\alpha(t) | t \in T\}$ and $\{\gamma(t) | t \in T\}$ as the mean-reversion level and the volatility of the short rate process, respectively. We assume that

$$\alpha(t) := \langle \alpha, X(t) \rangle,$$

and

$$\gamma(t) := \langle \gamma, X(t) \rangle,$$
where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^N$. Here, $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_N)' \in \mathbb{R}^N$ with $\alpha_i > 0$, and $\gamma := (\gamma_1, \gamma_2, \ldots, \gamma_N)' \in \mathbb{R}^N$ with $\gamma_i > 0$, for each $i = 1, 2, \ldots, N$. The mean-reversion coefficient $\beta$ describing the speed of mean-reversion is assumed to be a positive constant.

Let $\sigma(t)$ be the volatility of the underlying asset price at time $t$. Again we suppose that

$$\sigma(t) := \langle \sigma, X(t) \rangle,$$

where $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_N)' \in \mathbb{R}^N$ with $\sigma_i > 0$, for each $i = 1, 2, \ldots, N$.

Then under $\mathcal{P}$, we assume that the dynamics of the underlying asset value and the short rate are given by:

$$\begin{cases}
    dS(t) = r(t)S(t)dt + \sigma(t)S(t)dW_S(t), \\
    dr(t) = \beta(\alpha(t) - r(t))dt + \gamma(t)dW_r(t).
\end{cases}$$

where $W_S := \{W_S(t)| t \in T \}$ and $W_r := \{W_r(t)| t \in T \}$ are two standard Brownian motions with respect to their right-continuous, $\mathcal{P}$-completed, natural filtrations under $\mathcal{P}$. Furthermore, we suppose that the two Brownian motions $W_S$ and $W_r$ are correlated and the instantaneous correlation coefficient $\rho(t)$ at time $t$ is given by:

$$\langle W_S, W_r \rangle (t) = \int_0^t \rho(s)ds,$$

where $\rho(t) = \langle \rho, X(t) \rangle$ and $\rho := (\rho_1, \rho_2, \ldots, \rho_N)' \in \mathbb{R}^N$ with $-1 < \rho_i < 1$; $\{(W_S, W_r) (t)| t \in T \}$ is the (predictable) quadratic co-variation between $W_S$ and $W_r$.

Let $\mathcal{F}^X = \{\mathcal{F}^X(t)| t \in T \}$, $\mathcal{F}^S = \{\mathcal{F}^S(t)| t \in T \}$ and $\mathcal{F}^r = \{\mathcal{F}^r(t)| t \in T \}$ be the natural filtrations generated by $\{X(t)| t \in T \}$, $\{S(t)| t \in T \}$ and $\{r(t)| t \in T \}$ respectively. We assume that the filtrations given above are right continuous and $\mathcal{P}$-complete. We define two enlarged filtrations $\mathcal{G} := \{\mathcal{G}(t)| t \in T \}$ and $\mathcal{H} := \{\mathcal{H}(t)| t \in T \}$ respectively.
by letting

$$\mathcal{G}(t) := \mathcal{F}^{r}(t) \vee \mathcal{F}^{X}(t),$$

and

$$\mathcal{H}(t) := \mathcal{F}^{r}(t) \vee \mathcal{F}^{S}(t) \vee \mathcal{F}^{X}(t).$$

Here $\mathcal{A} \vee \mathcal{B}$ represents the minimal $\sigma$-field containing both the $\sigma-$fields $\mathcal{A}$ and $\mathcal{B}$.

### 4.3 Bond pricing and the forward measure

Shen and Siu (2013a) employed a partial differential equation approach to derive an exponential affine formula for the price of a zero-coupon bond and gave the forward measure when taking the zero-coupon bond as the numéraire. We shall use these results in our later developments. In this section, we shall present the results without giving the proofs.

Under the risk-neutral probability $\mathcal{P}$, the price at time $t$ of a zero-coupon bond, whose payoff is one unit of account at the maturity time $T$, is given by:

$$P(t, T) = \mathbb{E}\left[ \exp \left( -\int_{t}^{T} r(s) ds \right) \bigg| \mathcal{G}(t) \right]$$

$$= \mathbb{E}\left[ \exp \left( -\int_{t}^{T} r(s) ds \right) \bigg| r(t), X(t) \right]$$

Here $\mathbb{E}$ is an expectation under the measure $\mathcal{P}$. The last equality is attributed to the fact that $(r, X)$ is jointly Markovian with respect to the filtration $\mathcal{G}$. We write $P(t, T) := F(t, T, r(t), X(t)).$

The following lemma was given in Shen and Siu (2013a).
Lemma 4.3.1. Suppose \((t, r) \to F(t, T, r, x)\) is continuously differentiable w.r.t \(t\) and twice continuously differentiable w.r.t. \(r\), for each \(x \in \mathcal{E}\). Denote the corresponding partial derivatives by \(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial r}\) and \(\frac{\partial^2 F}{\partial r^2}\). Then, under the following regularity conditions,

1. \(E[\exp(-\int_0^T r(t)dt)] < \infty\);
2. \(E[\int_0^T |e^{-\int_0^t r(u)du} \frac{\partial F}{\partial r} \gamma(t)|^2 dt] < \infty\);

the bond price process has the following regime-switching exponential-affine representation:

\[
P(t, T) = \exp[A(t, T, X(t)) - B(t, T)r(t)] , \quad t \in T ,
\]

where \(A(t, T, X(t))\) and \(B(t, T)\) are some “smooth” functions satisfying

\[
A(t, T, X(t)) = \log \left\{ E \left[ \exp \left\{ \int_t^T \left( \alpha(s)\beta(s)B(s, T) - \frac{1}{2} \gamma^2(s)B^2(s, T) \right) ds \right\} | X(t) \right] \right\} ,
\]
\[
B(t, T) = \frac{1}{\beta} \left( 1 - e^{-\beta(T-t)} \right) , \quad t \in T .
\]

Let \(\tilde{A} := \tilde{A}(t, T, x) = \exp(A(t, T, x))\) and \(\tilde{A} := (\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_N)' \in \mathbb{R}^N\) with \(\tilde{A}_i := \tilde{A}(t, T, e_i)\), for each \(i = 1, 2, \cdots, N\). Then, the dynamics of the bond price process is given by

\[
\frac{dP(t, T)}{P(t, T)} = r(t)dt - \gamma(t)B(t, T)dW_r(t) + \tilde{A}(t, T, X(t))^{-1} \left\langle \tilde{A}(t, T), d\mathbf{M}(t) \right\rangle . \quad (4.3.1)
\]

The following lemma gives the dynamics of the underlying asset value, the interest rate and the Markov chain under a forward measure \(\mathcal{P}^T\) to be defined below.

Lemma 4.3.2. Let \(\Lambda(T)\) denote the Radon-Nikodym derivative defined by:

\[
\Lambda(T) = \left. \frac{d\mathcal{P}^T}{d\mathcal{P}} \right|_{G(T)} := \frac{\exp(-\int_0^T r(t)dt)}{E[\exp(-\int_0^T r(t)dt)]} . \quad (4.3.2)
\]

Under the following assumptions:
1. The Novikov condition is satisfied:

\[ E\left[ \exp\left\{ \frac{1}{2} \int_0^T \gamma^2(t)B^2(t,T)dt \right\} \right] < \infty . \]

2. \( \tilde{A}(t,T,x) \) is a suitable function in the sense that

\[ \frac{\tilde{A}(t,T,X(t))}{\tilde{A}(0,T,X(0))} \exp\left\{ -\int_0^t \frac{\partial \tilde{A}}{\partial s} + Q\tilde{A}(s,T,X(s))ds \right\} , \quad t \in \mathcal{T} , \]

is a \((\mathcal{G}, \mathcal{P})\)-martingale.

Then, under the forward probability measure \( \mathcal{P}^T \), the following results hold:

1. The dynamics of the underlying asset value and the short rate are given by:

\[
\begin{aligned}
\frac{dS(t)}{S(t)} &= \left( r(t) - \rho(t)\gamma(t)\sigma(t)B(t,T) \right) dt + \sigma(t)S(t)dW^T_S(t) , \\
\frac{dr(t)}{r(t)} &= \left( \beta_\alpha(t) - \gamma^2(t)B(t,T) - \beta r(t) \right) dt + \gamma(t)dW^T_r(t) ,
\end{aligned}
\]

(4.3.3)

where

\[ W^T_S(t) = W_S(t) + \int_0^t \rho(s)\gamma(s)B(s,T)ds , \quad t \in \mathcal{T} , \]

and

\[ W^T_r(t) = W_r(t) + \int_0^t \gamma(s)B(s,T)ds , \quad t \in \mathcal{T} , \]

are \( \mathcal{P}^T \)-standard Brownian motions with instantaneous correlation coefficient \( \rho(t) \) at time \( t \), i.e., \( \langle W^T_S, W^T_r \rangle (t) = \int_0^t \rho(s)ds \).

2. The rate matrix of the chain \( X \) is \( Q^T(t) := [q^T_{ij}(t)]_{i,j=1,2,...,N} \):

\[ q^T_{ij}(t) = \begin{cases} 
q_{ij} \frac{\tilde{A}(t,T,e_j)}{\tilde{A}(t,T,e_i)} , & i \neq j , \\
- \sum_{k \neq i} q_{ik} \frac{\tilde{A}(t,T,e_k)}{\tilde{A}(t,T,e_i)} , & i = j ,
\end{cases} \]

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and the semimartingale dynamics of the chain is given by:

\[ X(t) = X(0) + \int_0^t Q^T(s)X(s)ds + M^T(t), \quad t \in T, \]

where \( \{M^T(t) | t \in T\} \) is an \( \mathfrak{N} \)-valued, \( (\mathcal{F}^X, \mathcal{P}^T) \)-martingale.

Proof. Define

\[ \Lambda(t) := E[\Lambda(T)|\mathcal{G}(t)] = e^{-\int_0^t r(s)ds} \frac{P(t,T)}{P(0,T)}. \]

Then, from Eq. (4.3.1),

\[
\frac{d\Lambda(t)}{\Lambda(t)} = \frac{dP(t,T)}{P(t,T)} - r(t)dt
= -\gamma(t)B(t,T)dW_r(t) + \tilde{A}(t, T, X(t))^{-1} \left\langle \tilde{A}(t, T), dM(t) \right\rangle.
\]

Under the independence of \( W_r \) and \( M \), it is easy to see that \( \Lambda(t) = \Lambda_1(t) \cdot \Lambda_2(t) \), where

\[
\frac{d\Lambda_1(t)}{\Lambda_1(t)} = -\gamma(t)B(t, T)dW_r(t),
\]

and

\[
\frac{d\Lambda_2(t)}{\Lambda_2(t)} = \tilde{A}(t, T, X(t))^{-1} \left\langle \tilde{A}(t, T), dM(t) \right\rangle.
\]

Hence

\[
\Lambda_1(t) = \exp \left\{ -\int_0^t \gamma(s)B(s, T)dW_r(s) - \frac{1}{2} \int_0^t \gamma^2(s)B^2(s, T)ds \right\},
\]

and

\[
\Lambda_2(t) = \frac{\tilde{A}(t, T, X(t))}{A(0, T, X(0))} \exp \left\{ -\int_0^t \frac{\partial \tilde{A}}{\partial s} + Q\tilde{A}(s, T, X(s)) \right\} ds \right\}.
\]

From the assumptions, the \( (\mathcal{G}, \mathcal{P}) \)-local-martingales \( \{\Lambda(t) | t \in T\}, \{\Lambda_1(t) | t \in T\} \)
and \( \{\Lambda_2(t) | t \in T\} \) are also \( (\mathcal{G}, \mathcal{P}) \)-martingales. Therefore, \( \{\Lambda(t) | t \in T\} \) is a Radon-Nikodym derivative and the forward measure \( \mathcal{P}^T \) defined by Eq. (4.3.2) is a probability
measure. The Girsanov's theorem derives the result that \( W_T^S(t) \) and \( W_T^r(t) \) are \( \mathcal{P}^T \)-standard Brownian motions with correlation coefficient \( \rho(t) \). The second result follows from Lemma 12.3.3 in Rolski et al. (1999) or Proposition 5.1 in Palmowski and Rolski (2002).

### 4.4 Pricing the option using FFT methods

In this section, we derive the price of a European call option under the regime-switching stochastic interest rate model. Here, we adopt the Fourier methods, introduced in Carr and Madan (1999), to value the option.

Under the risk-neutral probability measure \( \mathcal{P} \), the price of a \( T \)-maturity option at time 0 is given by

\[
C(0, T) = E \left[ \exp \left( - \int_0^T r(t) dt \right) (S(T) - K)^+ \right],
\]

where \( S(T) \) is the underlying asset value at maturity \( T \) and \( K \) is the strike price. By change of measures defined in the earlier section, Eq. (4.4.1) becomes

\[
C(0, T) = P(0, T) E^T [(S(T) - K)^+] ,
\]

where \( E^T \) is an expectation under the forward measure \( \mathcal{P}^T \). To apply the Fourier transform, we first define the characteristic function of the logarithmic terminal price. Let \( s(T) \) denote the logarithm of the asset price at maturity time \( T \), i.e., \( s(T) = \ln S(T) \). By Eq. (4.3.3) and Itô's differentiation rule, under the forward measure \( \mathcal{P}^T \),

\[
\begin{cases}
    ds(t) = (r(t) - \frac{1}{2} \sigma^2(t) - \rho(t) \gamma(t) \sigma(t) B(t, T)) dt + \sigma(t) dW^T_S(t) , \\
    dr(t) = (\beta \alpha(t) - \gamma^2(t) B(t, T) - \beta r(t)) dt + \gamma(t) dW^T_r(t) .
\end{cases}
\]

\[
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\]
Lemma 4.4.1. For each $t \in \mathcal{T}$, let $\mathbf{J}(t,T) := (J_1(t,T), J_2(t,T), \ldots, J_N(t,T))$, where $J_i(t,T)$ is the occupation time of the chain $\mathbf{X}$ in state $e_i$ in the interval $[t,T]$, and $\phi(t) := (\phi_1(t), \phi_2(t), \ldots, \phi_N(t))$. $\Phi_{\mathbf{J}(t,T)|\mathcal{H}(t)}(\phi(t))$ is the conditional moment-generating function of the vector of occupation times $\mathbf{J}(t,T)$ given $\mathcal{H}(t)$ under $\mathcal{P}^T$ evaluated at $\phi(t)$. That is,

$$
\Phi_{\mathbf{J}(t,T)|\mathcal{H}(t)}(\phi(t)) := E^T[\exp(\langle \phi(t), \mathbf{J}(t,T) \rangle)|\mathcal{H}(t)] .
$$

(4.4.4)

Let $\text{diag}(\phi(t))$ denote the diagonal matrix with diagonal elements given by the components of $\phi(t)$. Then

$$
\Phi_{\mathbf{J}(t,T)|\mathcal{H}(t)}(\phi(t)) = \left\langle \mathbf{X}(t) \exp \left\{ \int_t^T (Q^T(s) + \text{diag}(\phi(s)))ds \right\}, 1 \right\rangle .
$$

(4.4.5)

Proof. The proof is a modification of that of Lemma 5.1 in Elliott and Siu (2013). Define, for each $t \in \mathcal{T}$, an $\mathbb{R}^N$-valued process $\{W(t,u)|u \in [t,T]\}$ by setting:

$$
W(t,u) := \exp(\langle \phi(t), \mathbf{J}(t,u) \rangle)X(u) = \exp \left( \int_t^u \langle \phi(s), X(s) \rangle ds \right)X(u) .
$$

Consequently,

$$
dW(t,u) = \langle \phi(u), X(u) \rangle W(t,u)du + \exp(\langle \phi(t), \mathbf{J}(t,u) \rangle)dX(u) .
$$

Note that under $\mathcal{P}^T$,

$$
dX(u) = Q^T(u)X(u-)du + dM^T(u) ,
$$

and that

$$
\langle \phi(u), X(u) \rangle W(t,u) = \text{diag}(\phi(u))W(t,u) , \quad t \in \mathcal{T} .
$$
Then,
\[
dW(t, u) = [Q^T(u) + \text{diag}(\phi(u))]W(t, u)du + \exp(\langle \phi(u), J(t, u) \rangle) dM^T(u).
\]

Consequently,
\[
W(t, u) = W(t, t) + \int_t^u (Q^T(s) + \text{diag}(\phi(s)))W(t, s)ds
+ \int_t^u \exp(\langle \phi(s), J(t, s) \rangle) dM^T(s)
= X(t) + \int_t^u (Q^T(s) + \text{diag}(\phi(s)))W(t, s)ds
+ \int_t^u \exp(\langle \phi(s), J(t, s) \rangle) dM^T(s).
\]

 Conditioning both sides on \( \mathcal{H}(t) \) under \( P^T \) gives
\[
E^T[W(t, u)|\mathcal{H}(t)] = X(t) + \int_t^u (Q^T(s) + \text{diag}(\phi(s)))E^T[W(t, s)|\mathcal{H}(t)]ds.
\]

Solving gives:
\[
E^T[W(t, u)|\mathcal{H}(t)] = X(t) \exp \left\{ \int_t^u (Q^T(s) + \text{diag}(\phi(s)))ds \right\}.
\]

Now,
\[
\Phi_{J(t,T)|\mathcal{H}(t)}(\phi(t)) = E^T[\exp(\langle \phi(t), J(t, T) \rangle)|\mathcal{H}(t)]
= E^T[\exp(\langle \phi(t), J(t, T) \rangle) \langle X(T), 1 \rangle |\mathcal{H}(t)]
= E^T[\langle \exp(\langle \phi(t), J(t, T) \rangle)X(T), 1 \rangle |\mathcal{H}(t)]
= \langle E^T[\exp(\langle \phi(t), J(t, T) \rangle)X(T)|\mathcal{H}(t)], 1 \rangle
= \langle E^T[W(t, T)|\mathcal{H}(t)], 1 \rangle
= \langle X(t) \exp \left\{ \int_t^T (Q^T(s) + \text{diag}(\phi(s)))ds \right\}, 1 \rangle.
\]
Lemma 4.4.2. Under the forward probability measure $\mathcal{P}^T$, the characteristic function of $\left( \int_0^T r(t) dt, \int_0^T \sigma(t) dW^T_S(t) \right)$ conditional on $\mathcal{F}^X(T)$ is given by:

\[
E^T \left[ \exp \left\{ iv \left( \int_0^T r(t) dt + \int_0^T \sigma(t) dW^T_S(t) \right) \right\} | \mathcal{F}^X(T) \right] = \exp \left\{ ivB(0,T)r(0) - \frac{1}{2} v^2 \int_0^T \sigma^2(u) du - \frac{1}{2} v^2 \int_0^T B(u,T)^2 \gamma^2(u) du \right. \\
+ iv \int_0^T B(u,T) [\beta \alpha(u) - \gamma^2(u)B(u,T)] du \\
- v^2 \int_0^T B(u,T) \rho(u) \sigma(u) \gamma(u) du \right\}.
\]

Proof. By Itô’s differentiation rule, we obtain the following equation:

\[
r(t) = e^{-\beta t} \left( r(0) + \int_0^t e^{\beta u} (\beta \alpha(u) - \gamma^2(u)B(u,T)) du + \int_0^t e^{\beta u} \gamma(u) dW^T_S(u) \right).
\]

Following Elliott and Kopp (2005), we can argue that conditional on $\mathcal{F}^X(T)$, $\int_0^T r(t) dt$ is normal with the following mean and variance

\[
E^T \left( \int_0^T r(t) dt | \mathcal{F}^X(T) \right) = \int_0^T e^{-\beta t} \left( r(0) + \int_0^t e^{\beta u} (\beta \alpha(u) - \gamma^2(u)B(u,T)) du \right) dt,
\]

\[
\text{Var}^T \left( \int_0^T r(t) dt | \mathcal{F}^X(T) \right) = \int_0^T e^{2\beta u} \gamma^2(u) \left( \int_u^T e^{-\beta s} ds \right)^2 du.
\]

Note that, under the forward measure $\mathcal{P}^T$, conditional on $\mathcal{F}^X(T)$, $\int_0^T \sigma(t) dW^T_S(t)$ is also normal with mean and variance given by 0 and $\int_0^T \sigma^2(t) dt$ respectively. Thus, the mean matrix and the covariance matrix of $\int_0^T r(t) dt$ and $\int_0^T \sigma(t) dW^T_S(t)$ conditional on $\mathcal{F}^X(T)$ are given by

\[
a = \left( \begin{array}{c} 0 \\
\int_0^T e^{-\beta t} \left( r(0) + \int_0^t e^{\beta u} [\beta \alpha(u) - \gamma^2(u)B(u,T)] du \right) dt \end{array} \right),
\]
and

\[
\sum = \begin{pmatrix}
\int_0^T \sigma^2(u)du & \int_0^T \rho(u)\sigma(u)e^{\beta u} \gamma(u) \left( \int_u^T e^{-\beta s} ds \right) du \\
\int_0^T \rho(u)\sigma(u)e^{\beta u} \gamma(u) \left( \int_u^T e^{-\beta s} ds \right) du & \int_0^T e^{2\beta u} \gamma^2(u) \left( \int_u^T e^{-\beta s} ds \right)^2 du
\end{pmatrix}.
\]

Thus, the characteristic function of \( \left( \int_0^T \sigma(t)dW_S(t), \int_0^T r(t)dt \right) \) conditional on \( \mathcal{F}^X(T) \) is

\[
\mathbb{E}^T \left[ \exp \left\{ iv \left( \int_0^T r(t)dt + \int_0^T \sigma(t)dW_S(t) \right) \right\} \Big| \mathcal{F}^X(T) \right]
\]

\[
= \exp \left\{ iv \int_0^T e^{-\beta t} \left( r(0) + \int_0^t e^{\beta u} [\beta \alpha(u) - \gamma^2(u) B(u,T)] du \right) dt \\
- \frac{1}{2} v^2 \int_0^T \sigma^2(u)du - \frac{1}{2} v^2 \int_0^T e^{2\beta u} \gamma^2(u) \left( \int_u^T e^{-\beta s} ds \right)^2 du \\
- v^2 \int_0^T \rho(u)\sigma(u)e^{\beta u} \gamma(u) \left( \int_u^T e^{-\beta s} ds \right) du \right\}
\]

\[
= \exp \left\{ iv \frac{1}{\beta} \left( 1 - e^{-\beta T} \right) r(0) - \frac{1}{2} v^2 \int_0^T \sigma^2(u)du \\
+ iv \int_0^T \frac{1}{\beta} \left( 1 - e^{-\beta(T-u)} \right) \left[ \beta \alpha(u) - \gamma^2(u) B(u,T) \right] du \\
- \frac{1}{2} v^2 \int_0^T \frac{1}{\beta^2} \left( 1 - e^{-\beta(T-u)} \right)^2 \gamma^2(u) du \\
- v^2 \int_0^T \frac{1}{\beta} \left( 1 - e^{-\beta(T-u)} \right) \rho(u)\sigma(u)\gamma(u) du \right\}.
\]

\[\square\]

**Proposition 4.4.1.** For each \( t \in \mathbb{R} \) and \( j = 1, 2, \ldots, N \), define

\[
\phi_j(t) := -iv \left( \frac{1}{2} \sigma_j^2 + \rho_j \gamma_j \sigma_j B(t,T) \right) - \frac{1}{2} v^2 \sigma_j^2 - \frac{1}{2} v^2 B(t,T)^2 \gamma_j^2 \\
+ iv B(t,T) (\beta \alpha_j - \gamma_j^2 B(t,T)) - v^2 B(t,T) \rho_j \sigma_j \gamma_j.
\]
Then, under the forward measure \( \mathcal{P}^T \), the characteristic function of \( s(T) \) is:

\[
\zeta(v, 0, T) := E_T[\exp \{iv s(T)\}]
= \exp \left\{ iv \left( s(0) + B(0,T) r(0) \right) \right\}
\langle X(0) \exp \left\{ \int_0^T (Q^T(t) + \text{diag}(\phi(t))) dt \right\}, 1 \rangle.
\]

(4.4.7)

**Proof.** Under the forward measure \( \mathcal{P}^T \), the characteristic function of \( s(T) \) is:

\[
\zeta(v, 0, T) = E_T[\exp \{iv s(T)\}]
= E_T \left[ E_T \left[ \exp \left\{ iv \left( s(0) + \int_0^T \left( r(t) - \frac{1}{2} \sigma^2(t) - \rho(t) \gamma(t) \sigma(t) B(t,T) \right) dt + \int_0^T \sigma(t) dW^T_S(t) \right) \right] F^X(T) \right] \]
= E_T \left[ \exp \left\{ iv \left( s(0) - \int_0^T \left( \frac{1}{2} \sigma^2(t) + \rho(t) \gamma(t) \sigma(t) B(t,T) \right) dt \right) \right] E_T \left[ \exp \left\{ iv \left( \int_0^T r(t) dt + \int_0^T \sigma(t) dW^T_S(t) \right) \right] F^X(T) \right] \]
= E_T \left[ \exp \left\{ iv \left( s(0) + \frac{1}{\beta} \left( 1 - e^{-\beta T} \right) r(0) \right) + \sum_{j=1}^N \phi_j(t) J_j(0,T) \right\} \right]
= E_T \left[ \exp \left\{ iv \left( s(0) + \frac{1}{\beta} \left( 1 - e^{-\beta T} \right) r(0) \right) + \langle \phi(t), J(0,T) \rangle \right\} \right]
= \exp \left\{ iv \left( s(0) + \frac{1}{\beta} \left( 1 - e^{-\beta T} \right) r(0) \right) \right\}
\langle X(0) \exp \left\{ \int_0^T (Q^T(t) + \text{diag}(\phi(t))) dt \right\}, 1 \rangle.
\]

Following the notation in Carr and Madan (1999) and Schrager and Pelsser (2006), write \( k = \ln(K) \), the dampened call price is given by

\[
c(k) := \exp(ak) C(0, T),
\]
where $\alpha$ is called the dampening coefficient and assumed to be positive\(^2\). Then we can get the dampened call transform

$$\xi(u, 0, T) = \int_{\mathbb{R}} \exp(iuk)c(k)dk .$$

Let $f_{\mathcal{F}^X}(s)$ denote the conditional density function of $s(T)$ given $\mathcal{F}^X(T)$. Then,

$$\xi(u, 0, T) = \int_{\mathbb{R}} e^{iuk}c(k)dk \hspace{1cm} = \int_{\mathbb{R}} e^{iuk} e^{\alpha k} P(0, T) E^T[(e^{s(T)} - e^k)_+]dk$$

$$= P(0, T) E^T\left[\int_{\mathbb{R}} e^{iuk} e^{\alpha k} E^T[(e^{s(T)} - e^k)_+|\mathcal{F}^X(T)]dk\right]$$

$$= P(0, T) E^T\left[\int_{\mathbb{R}} e^{iuk} e^{\alpha k} \int_{k}^{\infty} (e^s - e^k)f_{\mathcal{F}^X}(s)dsdk\right]$$

$$= P(0, T) E^T\left[\int_{\mathbb{R}} f_{\mathcal{F}^X}(s) \int_{-\infty}^{s} (e^{s} e^{(\alpha+iu)k} - e^{(1+\alpha+iu)k})dkds\right]$$

$$= P(0, T) E^T\left[\int_{\mathbb{R}} f_{\mathcal{F}^X}(s) \left(\frac{e^{(1+\alpha+iu)s}}{\alpha + iu} - \frac{e^{(1+\alpha+iu)s}}{1 + \alpha + iu}\right)ds\right]$$

$$= \frac{P(0, T) \zeta(u - i(\alpha + 1), 0, T)}{\alpha^2 + \alpha - u^2 + i(2\alpha + 1)u} . \quad (4.4.8)$$

Then, as in Carr and Madan (1999) and Schrager and Pelsser (2006), the pricing formula of the call option can be written as:

$$c(k) = \frac{\exp(-\alpha k)}{\pi} \int_{0}^{\infty} [\exp(-iuk)\xi(u, 0, T)]du . \quad (4.4.9)$$

### 4.5 Numerical examples

In this section, we illustrate the option valuation adopting the FFT approach. Here, without of loss of generality, we assume there are only two states of the continuous-\(^2\) To obtain a square integrable function, the dampening coefficient $\alpha$ is selected and the dampened call pricing formula is defined. The problem how to choose the value of the coefficient $\alpha$ is considered in Carr and Madan (1999).
time, finite-state Markov chain $X$, i.e., State 1 and State 2, representing a ‘Good’ economy and a ‘Bad’ one, respectively. Write $X(t) = (1, 0)'$ and $X(t) = (0, 1)'$, $\forall t \in T$, for State 1 and State 2. Then, the generator of the chain $X$ under $\mathcal{P}$ is given by

$$Q = \begin{pmatrix} -q & q \\ q & -q \end{pmatrix},$$

where $q$ takes values in $[0,1]$. Note that the rate matrix $Q$ is symmetric, which means the probabilities of the chain transiting from State 1 (State 2) to State 2 (State 1) within a fixed period are equal. The parameters are assumed to be the following values:

$$\alpha = (\alpha_1, \alpha_2)' = (0.04, 0.02)', \quad \gamma = (\gamma_1, \gamma_2)' = (0.02, 0.04)',$$

$$\sigma = (\sigma_1, \sigma_2)' = (0.2, 0.4)', \quad \rho = (\rho_1, \rho_2)' = (-0.5, -0.75)'.$$

For simplicity, we assume the constant mean reversion coefficient is $\beta = 0.2$.

Then our two-state model becomes

$$\begin{cases} dS(t) = r(t)S(t)dt + \sigma_1 S(t)dW_S(t) \\ dr(t) = \beta(\alpha_1 - r(t))dt + \gamma_1 dW_r(t) \end{cases}, \quad \text{if } X(t) = (1, 0)', \quad (4.5.1)$$

and

$$\begin{cases} dS(t) = r(t)S(t)dt + \sigma_2 S(t)dW_S(t) \\ dr(t) = \beta(\alpha_2 - r(t))dt + \gamma_2 dW_r(t) \end{cases}, \quad \text{if } X(t) = (0, 1)', \quad (4.5.2)$$

where the instantaneous correlation coefficient of $W_r$ and $W_S$ in State $i$ is assumed to be $\rho_i$, $i = 1, 2$. The initial values of underlying asset and interest rate are assumed to be $S(0) = 0.98$ and $r(0) = 0.02$.

For our FFT method, we assume the grid size of $u$ is 0.2441 in our quadrature. From Lemma 4.3.1, Lemma 4.3.2 and Proposition 4.4.1, we price the European call option with a maturity $T = 1$ under our model.
Table 4.5.1: Prices calculated by FFT when $q = 0.5$ and $T = 1$

<table>
<thead>
<tr>
<th>Strikes</th>
<th>$N = 512$</th>
<th>$N = 1024$</th>
<th>$N = 2048$</th>
<th>$N = 4096$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$</td>
<td>state 1</td>
<td>state 2</td>
<td>state 1</td>
<td>state 2</td>
</tr>
<tr>
<td>0.5202</td>
<td>1.2313</td>
<td>1.2327</td>
<td>1.2316</td>
<td>1.2330</td>
</tr>
<tr>
<td>0.5471</td>
<td>1.1206</td>
<td>1.1218</td>
<td>1.1208</td>
<td>1.1221</td>
</tr>
<tr>
<td>0.5753</td>
<td>1.0051</td>
<td>1.0063</td>
<td>1.0053</td>
<td>1.0065</td>
</tr>
<tr>
<td>0.6049</td>
<td>0.8865</td>
<td>0.8875</td>
<td>0.8867</td>
<td>0.8877</td>
</tr>
<tr>
<td>0.6361</td>
<td>0.7663</td>
<td>0.7673</td>
<td>0.7665</td>
<td>0.7674</td>
</tr>
<tr>
<td>0.6689</td>
<td>0.6460</td>
<td>0.6469</td>
<td>0.6462</td>
<td>0.6469</td>
</tr>
<tr>
<td>0.7034</td>
<td>0.5271</td>
<td>0.5278</td>
<td>0.5272</td>
<td>0.5278</td>
</tr>
<tr>
<td>0.7396</td>
<td>0.4109</td>
<td>0.4115</td>
<td>0.4109</td>
<td>0.4115</td>
</tr>
<tr>
<td>0.7778</td>
<td>0.2986</td>
<td>0.2991</td>
<td>0.2986</td>
<td>0.2990</td>
</tr>
<tr>
<td>0.8179</td>
<td>0.1914</td>
<td>0.1918</td>
<td>0.1913</td>
<td>0.1916</td>
</tr>
<tr>
<td>0.8600</td>
<td>0.0903</td>
<td>0.0906</td>
<td>0.0902</td>
<td>0.0904</td>
</tr>
</tbody>
</table>
Table 4.5.1 illustrates the European option prices across a set of strikes under our model. For the same strike, the price of a European option in State 1 is systematically lower than that in State 2. This makes intuitive sense. The main feature of State 1 is a higher interest rate and a lower volatility, while the interest rate is lower and volatility is higher in State 2. It is not unreasonable that the price of the European option in State 1 (a “Good” economy) is lower than that in State 2 (a “Bad” economy) since an additional amount of risk premium may be required to compensate for a “Bad” economic condition. On the other hand, the option prices increase with the discretization number $N$ before converging. It is clear that the convergence rate of the FFT is comparatively fast from Table 4.5.1.

![Prices of European options with strike K=0.5](image1)

![Prices of European options with strike K=0.7](image2)

**Figure 4.5.1:** European option values corresponding to different $q$

**Figure 4.5.2:** European option values corresponding to different $q$

From Figs. 4.5.1 and 4.5.2, assuming $T = 1$ and $N = 4096$, we notice that the price of a European option increases with $q$ in State 1 while decreases with $q$ in State 2. When $q$ increases, the probability of the chain $X$ transiting from State 1 to State 2 will increase. As explained earlier, the European options are cheaper in State 1. Thus, the price of the European options at State 1 (State 2) increases (decreases) with the
probability of the chain transiting from State 1 to State 2. This is the reason why the European options are more expensive when \( q \) increases (decreases) in State 1 (State 2). Note that the probability of the chain transiting from State 1 (State 2) to State 2 (State 1) is zero when \( q = 0 \). Under this special case, the regime-switching effect does not exist.

Furthermore, we assume \( q = 0.5, N = 4096 \) and \( T = 1 \) and illustrate the European option prices under our model with an initial asset value \( S(0) \) and a strike value \( K \) in both State 1 and State 2. When \( q \) and \( T \) remain the same, Figs. 4.5.3 and 4.5.4 illustrate that the prices of the options decrease with \( K \) and increase with \( S(0) \) in both State 1 and State 2.

![Figure 4.5.3: European option values with different asset value \( S \) and strike \( K \) in State 1](image1)

![Figure 4.5.4: European option values with different asset value \( S \) and strike \( K \) in State 2](image2)

4.6 Conclusions

The fast Fourier transform (FFT) is a popular method to value options. In this paper, the FFT approach was used to price European options under a stochastic interest rate
model with regime-switching. Incorporating both interest rate and regime-switching risks, we first derived the characteristic function of the logarithmic terminal asset price under the forward measure. Once the analytical form of the characteristic function is obtained, the FFT was used to calculate the option prices numerically. Both a theoretical analysis and numerical examples illustrate the practicality of the proposed method.
Chapter 5

Pricing annuity guarantees under a double regime-switching model

5.1 Introduction

Equity-linked annuities (ELAs) are one of the major innovations in the insurance industry. They provide policyholders with insurance protection as well as investment returns from equity markets. These contracts allow the flexibility to provide both life insurance benefits and guaranteed minimum accumulation benefits. Typically, in an ELA contract, an insurer will make periodic payments to the beneficiary, while the policyholder pays a lump-sum premium at the initiation of the contract. From a policyholder’s perspective, ELAs provide minimum guarantees on downside risk and upside potential profits. The policyholder is also provided with the flexibility to select the composition of an investment portfolio. Furthermore, the tax-deferred feature is another advantage of these products. From the perspective of an insurer, higher insurance fees is a main advantage. In practice, the operational procedure to sell ELAs are
comparatively easier. These may explain why both policyholders and insurers prefer these products to other long-term investments with lower yields, including bank accounts, bonds, and so on. Two popular types of ELAs are equity-indexed annuities (EIAs) and variable annuities (VAs) with various embedded guarantees.

The valuation of equity-linked annuities (ELAs), including EIAs and VAs have attracted a considerable interest from both academic researchers and market practitioners. The literature mostly investigate the valuation of ELAs based on the interplay between an option and an ELA, (see Boyle and Schwartz (1977) and Brennan and Schwartz (1976, 1979)). The guaranteed minimum benefit can be viewed as a kind of embedded options. Much attention has been given to the EIAs valuation under the Black-Scholes framework, including Tiong (2000), Lee (2003), etc. Lin and Tan (2003) and Kijima and Wong (2007) investigated the valuation of EIAs with stochastic interest rates and mortality risk, while Qian et al. (2010) considered the EIAs valuation with stochastic mortality rate. Milevsky and Posner (2001) investigated the valuation of guaranteed minimum death benefit (GMDB) in VAs by the risk-neutral pricing theory. Examples of considering the valuation of guaranteed minimum withdrawal benefit (GMWB) in VAs include Milevsky and Salisbury (2006) and Dai et al. (2008). Hardy (2003) presented an overview of various investment guarantees. Bauer et al. (2008) considered a general pricing framework for all types of guarantees in VAs. Siu et al. (2007) and Ng et al. (2011) discussed the valuation of investment guarantees under GARCH-type models.

Regime-switching models are popular and practically useful models in econometrics and finance. This class of models was popularized by Hamilton (1989) to economics. One of the main advantages of regime-switching models is that they allow the flexibility to describe the impacts of the structural changes in economic conditions. Typically, the model dynamics are assumed to change over time with the states of a
modulating Markov chain, representing the states of an underlying economy. Recently, regime-switching models have been applied to various practical problems in finance and insurance. A major area of these applications is option valuation and its counterpart in insurance, namely fair valuation of insurance products. Due to the very nature of the ELAs such as the long-term maturity, the use of regime-switching models to evaluate fair values of liabilities underlying ELAs is deemed to be practically relevant. Hardy (2001) discussed the valuation and hedging of long-term investment guarantees under regime-switching models. Some examples of pricing participating life insurance products under regime-switching models include Siu (2005) and Siu et al. (2008a). Lin et al. (2009) discussed the valuation of EIAs and VAs under a regime-switching model under the assumption that the model dynamics of the reference investment fund value is a geometric Brownian motion modulated by a continuous-time, finite-state Markov chain. Yuen and Yang (2010b) applied the trinomial tree method to value EIAs with regime-switching. Ng and Li (2011) first studied the valuation of variable annuity guarantees under a multivariate regime-switching model. Qian et al. (2012) considered the valuation of equity-indexed annuities with regime-switching jump-diffusion model and stochastic mortality, where the jump-component is described by a compound Poisson process. Fan (2013) considered the valuation of variable annuity with GMDB when the investment fund follows a Markov-modulated geometric Brownian motion.

However, most of the existing literature on financial products valuation under regime-switching models suppose that only the model parameters change according to the modulating Markov chain. Comparatively little attention has been paid to regime-switching models with the assumption that both model parameters and the price level of the reference investment fund may change when the modulating Markov chain states switch. Using the terminology in Shen et al. (2014), when a regime switch occurs, the model with only model parameters changing is called the single regime-
switching (SRS) model, while the other kind of regime-switching model is denoted as the double regime-switching (DRS) model. The DRS models are more flexible than their SRS counterparts to describe the stochastic movements of the reference investment fund due to the fact that a jump in the investment fund price level occurs in the former, rather than the latter, when there is a regime switches (see Naik, 1993; Yuen and Yang, 2009; Elliott and Siu, 2011; Elliott et al., 2011b; Shen et al., 2014).

In this paper, we adopt the DRS model proposed in Shen et al. (2014) to investigate the valuation of the equity-linked products with mortality risk. Various designs including the point-to-point EIAs, the annual ratchet EIAs and the GMDB in variable annuities are considered. The main feature of the DRS model is that it provides a way to endogenously determine the regime-switching risk, while the regime-switching risk is either ignored or taken exogenously in earlier works. More specifically, the model parameters, including the risk-free interest rate, the appreciation rate and the volatility, are modulated by a continuous-time, finite-state, observable Markov chain. Meanwhile, the price level of the reference investment fund will have a jump when a regime switch occurs. The martingale, associated with the modulating Markov chain, is used to describe the jump component of the reference investment fund value. This describes quantitatively how jumps in the price level when regime switches occur. Here, we present two approaches selecting a pricing kernel. First, we use the generalized version of the regime-switching Esscher transform introduced in Shen et al. (2014) to select an equivalent martingale measure. Then we discuss the selection of the equivalent martingale measure using the minimal martingale measure. Both approaches allow us to incorporate not only the diffusion risk described by the Brownian motion but the regime-switching risk modeled by the Markov chain in the valuation. Under the selected risk-neutral probability measure, we use the inverse Fourier transform to derive integral pricing formulae for the embedded options. The fast Fourier transform
(FFT) method is adopted to discretize the integral pricing formulae. Since the double regime-switching model is an extension of the single one, the valuation problem under the single model considered in Fan (2013) may be considered a particular case of the valuation problem in our current paper. Using the FFT method, we provide the numerical examples to illustrate the valuation of the point-to-point EIAs and GMDB in VAs under both the double regime-switching model and the single regime-switching model as well as document the pricing implications of these two models.

Lin et al. (2009), one of our main references, considered an interesting problem to price annuity guarantees under a regime-switching model. Our paper extends the results of Lin et al. (2009) in the following aspects. Firstly, we consider the valuation of EIAs and VAs under a double regime-switching model in Shen et al. (2014). In addition to the assumption that model parameters are governed by the modulating Markov chain adopted in Lin et al. (2009), we also assume that a jump in the price level of the reference investment fund will occur when the modulating Markov chain switches from one state to another. In other words, the impacts of the regime-switching risk were not considered in Lin et al. (2009). However, the regime-switching risk brought by the state transitions of the underlying economy is difficult, if not impossible, to be diversified. This may suggest that the regime-switching risk may not be ignored. Secondly, we provide two different ways to endogenously determine the regime-switching risk using the generalized regime-switching Esscher transform and the minimal martingale measure approach. In the discussion part of Lin et al. (2009) by Elliott and Siu (2009), an exogenous way to quantify the regime-switching risk is provided. However, there may exist more than one solutions for the regime-switching Esscher transform parameters from the given density process described in the discussion of Lin et al. (2009). Other techniques are needed to choose an equivalent martingale measure. In the model we considered here without imposing other criteria or constraints, a unique
pricing kernel can be selected using either the generalized version of regime-switching Esscher transformation or the minimal martingale measure approach. Furthermore, this pricing kernel also provides a quantification for the regime-switching risk. Thirdly, our results may be easier to be extended to a multi-regime case. The analytical pricing formulae, obtained via the fast Fourier transform (FFT) approach, look quite neat and the convergence rate of the FFT is reasonably fast.

The rest of the paper is organized as follows. The next section presents the model dynamics. In Section 5.3, we select equivalent martingale measures using the generalized version of the regime-switching Esscher transformation and the minimal martingale measure approach. Section 5.4 presents the valuation of the point-to-point EIAs and the annual ratchet EIAs. The FFT approach and Monte Carlo method are applied to calculate the prices of the point-to-point EIAs and the annual ratchet EIAs, respectively. The valuation of the variable annuities with GMDB are considered in Section 5.5. In Section 5.6, we give numerical examples to illustrate the valuation of the point-to-point EIAs, the annual ratchet EIAs and VAs with GMDB. Section 5.7 concludes the paper.

5.2 The model dynamics

The modelling framework presented in this section resembles to that considered in Shen et al. (2014). Consider a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, under which describes uncertainties attributed to a standard Brownian motion and a Markov chain. We equip the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a filtration $\mathbb{F} := \{\mathcal{F}(t) | t \in \mathcal{T}\}$ satisfying the usual conditions of right-continuity and $\mathcal{P}$-completeness. Suppose that $\mathcal{P}$ is a real-world probability measure. Let $\mathcal{T}$ denote the time index set $[0, T]$ of the model, where $T < \infty$. We describe the evolution of the state of an economy over time by a
continuous-time, finite-state, observable Markov chain \( X := \{ X(t) \mid t \in T \} \) on \( (\Omega, \mathcal{F}, \mathcal{P}) \) taking values in a finite-state space \( S := \{ s_1, s_2, \ldots, s_N \} \). The states of the chain \( X \) are interpreted as different states of an economy or different stages of a business cycle. Without loss of generality, we adopt the canonical state space representation of the chain in Elliott et al. (1994) and identify the states of the chain with a finite set of standard unit vectors \( E := \{ e_1, e_2, \ldots, e_N \} \subset \mathbb{R}^N \), where the \( l \)th component of \( e_j \) is the Kronecker delta \( \delta_{jl} \), for each \( j, l = 1, 2, \ldots, N \).

Define \( A := [a_{jl}]_{j,l=1,2,\ldots,N} \) as the rate matrix of the chain \( X \) under \( \mathcal{P} \), where \( a_{jl} \) represents the transition intensity of the chain \( X \) from state \( e_j \) to state \( e_l \). Note that \( a_{jl} \geq 0 \), for \( j \neq l \) and \( \sum_{l=1}^{N} a_{jl} = 0 \) for each \( j, l = 1, 2, \ldots, N \). Then, the following semi-martingale representation is obtained by Elliott et al. (1994):

\[
X(t) = X(0) + \int_0^t A X(s) ds + M(t), \quad t \in T,
\]

where \( \{ M(t) \mid t \in T \} \) is an \( \mathbb{R}^N \)-valued square-integrable martingale with respect to the filtration generated by the Markov chain \( X \).

Suppose \( J_{jl}(t) \) is the number of jumps of the chain \( X \) from state \( e_j \) to state \( e_l \) up to time \( t \) and \( \Phi_l(t) \) counts the number of jumps of the chain \( X \) into state \( e_l \) from other states up to time \( t \) for each \( t \in T \) and \( j, l = 1, 2, \ldots, N \). Denote

\[
\Phi_l(t) := \sum_{j=1, j \neq l}^{N} J_{jl}(t) = \sum_{j=1, j \neq l}^{N} \sum_{0 < s \leq t} \langle X(s-), e_j \rangle \langle X(s), e_l \rangle
\]

\[\text{1}\]The Kronecker product \( \delta_{jl} \) is defined as:

\[
\delta_{jl} := \begin{cases} 
1, & \text{if } j = l, \\
0, & \text{otherwise}.
\end{cases}
\]
\[
= \sum_{j=1, j\neq l}^{N} \left[ \int_{0}^{t} \langle X(s-), e_j \rangle \langle AX(s), e_l \rangle \, ds + \int_{0}^{t} \langle X(s-), e_j \rangle \langle dM(s), e_l \rangle \right] \\
= \phi_l(t) + \Phi_l(t) ,
\]

where

\[
\phi_l(t) := \sum_{j=1, j\neq l}^{N} a_{jl} \int_{0}^{t} \langle X(s-), e_j \rangle \, ds ,
\]

and

\[
\Phi_l(t) := \sum_{j=1, j\neq l}^{N} \int_{0}^{t} \langle X(s-), e_j \rangle \langle dM(s), e_l \rangle .
\]

It is easy to see that \( \Phi_l(t) \) is a martingale. The differential form of the martingale \( \Phi_l(t) \) can be represented as

\[
d\Phi_l(t) = d\Phi_l(t) - a_l(t-)dt ,
\]

where \( a_l(t) := \sum_{j=1, j\neq l}^{N} a_{jl} \langle X(t), e_j \rangle \), for \( l = 1, 2, \ldots, N \). This is a version of the Doob-Meyer decomposition.

We assume that there are two primitive assets, namely, a zero-coupon bond \( B \) and an investment fund \( S \), in the financial market. The instantaneous market interest rate is supposed to be modulated by the chain \( X \) as follows:

\[
r(t) := \langle r, X(t) \rangle , \quad t \in T ,
\]

where \( r := (r_1, r_2, \ldots, r_N)' \in \mathbb{R}^N \), with \( r_j > 0 \) for each \( j = 1, 2, \ldots, N \). \( y' \) is the transpose of a vector or a matrix \( y \) and \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( \mathbb{R}^N \). Then the dynamics of the zero-coupon bond \( B := \{ B(t) | t \in T \} \) is given by

\[
 dB(t) = r(t)B(t)dt, \quad B(0) = 1 .
\]
Further, suppose the appreciation rate and the volatility of the investment fund at time $t$ are also modulated by the Markov chain $X$, i.e.,

$$
\mu(t) := \langle \mu, X(t) \rangle, \quad \sigma(t) := \langle \sigma, X(t) \rangle, \quad t \in \mathcal{T},
$$

where $\mu := (\mu_1, \mu_2, \ldots, \mu_N)' \in \mathbb{R}^N$ and $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_N)' \in \mathbb{R}^N$ for each $j = 1, 2, \ldots, N$.

Let $W := \{W(t) | t \in \mathcal{T}\}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$. To simplify our discussion, we suppose that $W$ and $X$ are stochastically independent under $\mathcal{P}$. Then, under the real-world probability measure $\mathcal{P}$, the price process of the investment fund is governed by the following double regime-switching model:

$$
\frac{dS(t)}{S(t)} = \mu(t-)dt + \sigma(t-)dW(t) + \sum_{l=1}^{N} (e^{\beta_l(t-)} - 1) d\Phi_l(t), \quad S(0) = S_0 > 0. \quad (5.2.1)
$$

where $\beta_l(t) := \langle \beta_l, X(t) \rangle$ and $\beta_i := (\beta_{1i}, \beta_{2i}, \ldots, \beta_{Ni})' \in \mathbb{R}^N$. Write $\beta := (\beta_1, \beta_2, \ldots, \beta_N) \in \mathbb{R}^{N \times N}$. Here the jump ratio in the investment fund price level when the chain transits from state $e_j$ to state $e_l$ is given by

$$
\begin{cases}
  e^{\beta_l} - 1, & j \neq l, \\
  0, & j = l.
\end{cases}
$$

Define $Y(t) = \log(S(t)/S_0)$ be the logarithmic return of the investment fund during the time horizon $[0, t]$, for each $t \in \mathcal{T}$. Then by Itô's differentiation rule, it is easy to see that

$$
dY(t) = \left[ \mu(t-) - \frac{1}{2} \sigma^2(t-) - \sum_{l=1}^{N} (e^{\beta_l(t-)} - 1) \beta_l(t-) \right] dt
$$

$$
+ \sigma(t-)dW(t) + \sum_{l=1}^{N} \beta_l(t-)d\Phi_l(t), \quad t \in \mathcal{T}. \quad (5.2.2)
$$

Write $Y := \{Y(t) | t \in \mathcal{T}\}$. Let $\mathcal{F}^X := \{\mathcal{F}^X(t) | t \in \mathcal{T}\}$, $\mathcal{F}^S := \{\mathcal{F}^S(t) | t \in \mathcal{T}\}$ and $\mathcal{F}^Y := \{\mathcal{F}^Y(t) | t \in \mathcal{T}\}$ be the right-continuous, $\mathcal{P}$-complete, natural filtrations generated
by the chain \( X \), the processes \( S \) and \( Y \), respectively. Since \( \mathbb{F}^S \) and \( \mathbb{F}^Y \) contain the same information, we can use either one of them as an observed information structure. Define the filtration \( \mathcal{G} = \{ \mathcal{G}(t) | t \in \mathcal{T} \} \) by setting \( \mathcal{G}(t) := \mathcal{F}^Y(t) \cup \mathcal{F}^X(t) \), the minimal \( \sigma \)-field containing \( \mathcal{F}^Y(t) \) and \( \mathcal{F}^X(t) \), for each \( t \in \mathcal{T} \).

5.3 Equivalent martingale measures

It is well known that there exist more than one equivalent martingale measures in the incomplete market. Various approaches have been proposed in the literature for the selection of an equivalent martingale measure in an incomplete market. In this section, we apply the generalized regime-switching Esscher transform and the minimal martingale measure method to determine an equivalent martingale measure. We present the (local)-martingale conditions in the two cases.

5.3.1 Esscher transformed equivalent martingale measure

In this subsection, we discuss how to select a pricing kernel using a generalized version of regime-switching Esscher transform. Let \( \mathcal{L}(Y) \) be the space of all processes \( \theta := \{ \theta(t) | t \in \mathcal{T} \} \) such that

1. For each \( t \in \mathcal{T} \), \( \theta(t) := \langle \theta, X(t) \rangle \), where \( \theta := (\theta_1, \theta_2, \ldots, \theta_N)' \in \mathbb{R}^N \);

2. \( \theta \) is integrable with respect to \( Y \) in the sense of stochastic integration.

For each \( \theta \in \mathcal{L}(Y) \), we denote the stochastic integral of \( \theta \) with respect to \( Y \) as

\[
(\theta \cdot Y)(t) := \int_0^t \theta(s) dY(s) , \quad t \in \mathcal{T},
\]

where \( \theta \) is called the Esscher transform parameter.
For each $\theta \in \mathcal{L}(Y)$, define a $\mathbb{G}$-adapted exponential process $D^\theta := \{D^\theta(t)|t \in \mathcal{T}\}$ as follows:

$$D^\theta(t) := \exp((\theta \cdot Y)(t)) = 1 + \int_0^t D^\theta(s-)dH^\theta(s),$$

where

$$H^\theta(t) := \int_0^t \theta(s) \left[ \mu(s) - \frac{1}{2} \sigma^2(s) - \sum_{l=1}^N \left( e^{\beta_l(s)} - 1 - \beta_l(s) \right) a_l(s) \right] ds$$

$$+ \int_0^t \frac{1}{2} \theta^2(s) \sigma^2(s) ds + \int_0^t \sum_{l=1}^N \left( e^{\theta(s)\beta_l(s)} - 1 - \theta(s)\beta_l(s) \right) a_l(s) ds$$

$$+ \int_0^t \theta(s) \sigma(s)dW(s) + \int_0^t \sum_{l=1}^N \left( e^{\theta(s)\beta_l(s)} - 1 \right) d\tilde{\Phi}_l(s),$$

is a $\mathbb{G}$-adapted process. Then $D^\theta$ is the Doléans-Dade stochastic exponential of $H^\theta$, i.e.

$$D^\theta(t) = \mathcal{E}(H^\theta(t)), \quad t \in \mathcal{T}.$$ 

The Laplace cumulant process $^2$ of the stochastic integral process $(\theta \cdot Y)$ under $\mathcal{P}$ is given by

$$\mathcal{M}^\theta(t) = \int_0^t \theta(s) \left[ \mu(s) - \frac{1}{2} \sigma^2(s) - \sum_{l=1}^N \left( e^{\beta_l(s)} - 1 - \beta_l(s) \right) a_l(s) \right] ds$$

$$+ \int_0^t \frac{1}{2} \theta^2(s) \sigma^2(s) ds + \int_0^t \sum_{l=1}^N \left( e^{\theta(s)\beta_l(s)} - 1 - \theta(s)\beta_l(s) \right) a_l(s) ds (5.3.1)$$

This is the predictable finite-variation part of $H^\theta$.

---

$^2$For more discussion on the Laplace cumulant process as well as the Esscher transform given below, interested readers can refer to Kallsen and Shiryaev (2002) and Elliott and Siu (2013).
The Doléans-Dade exponential $\mathcal{E}(\mathcal{M}^\theta(t))$ of $\mathcal{M}^\theta(t)$ is then the unique solution of the equation:

$$
\mathcal{E}(\mathcal{M}^\theta(t)) = 1 + \int_0^t \mathcal{E}(\mathcal{M}^\theta(s))d\mathcal{M}^\theta(s)
= \exp(\mathcal{M}^\theta(t)), \quad t \in \mathcal{T}.
$$

The second equality is due to the fact that $\{\mathcal{M}^\theta(t)|t \in \mathcal{T}\}$ is a finite variation process.

Then, the logarithmic transform $\tilde{\mathcal{M}}^\theta := \{\tilde{\mathcal{M}}^\theta(t)|t \in \mathcal{T}\}$ of $\mathcal{M}^\theta(t)$, for each $\theta \in \mathcal{L}(Y)$, is given by

$$
\tilde{\mathcal{M}}^\theta(t) := \log(\mathcal{E}(\mathcal{M}^\theta(t))) = \mathcal{M}^\theta(t), \quad t \in \mathcal{T}.
$$

(5.3.2)

Let $\Lambda^\theta := \{\Lambda^\theta(t)|t \in \mathcal{T}\}$ be a $\mathcal{G}$-adapted process associated with $\theta \in \mathcal{L}(Y)$ defined by:

$$
\Lambda^\theta(t) := \exp \left( (\theta \cdot Y)(t) - \tilde{\mathcal{M}}^\theta(t) \right), \quad t \in \mathcal{T}.
$$

Then from Eqs. (5.3.1) and (5.3.2), we obtain

$$
\Lambda^\theta(t) = \exp \left\{ \int_0^t \theta(s)\sigma(s)dW(s) - \frac{1}{2} \int_0^t \theta^2(s)\sigma^2(s)ds + \int_0^t \sum_{l=1}^N \theta(s)\beta_l(s)d\tilde{\Phi}_l(s) \\
- \int_0^t \sum_{l=1}^N \left[ e^{\theta(s)\beta_l(s)} - 1 - \theta(s)\beta_l(s) \right] a_l(s)dt \right\}.
$$

(5.3.3)

Since $\Lambda^\theta$ is an exponential process compensated by its modified Laplace cumulant process, it is a $(\mathcal{G}, \mathcal{P})$-(local)-martingale. Indeed, this can also be checked easily via Itô’s differentiation rule. It is assumed that $\theta \in \mathcal{L}(Y)$ satisfying certain standard technical conditions so that $\Lambda^\theta$ is a $(\mathcal{G}, \mathcal{P})$-martingale.

For each $\theta \in \mathcal{L}(Y)$, we define a new probability measure $\mathcal{Q}^\theta$ equivalent to $\mathcal{P}$ on $\mathcal{G}(T)$ by a generalized version of the regime-switching Esscher transform $\Lambda^\theta(T)$ as follows:

$$
\frac{d\mathcal{Q}^\theta}{d\mathcal{P}} \bigg|_{\mathcal{G}(T)} := \Lambda^\theta(T).
$$
The following lemmas, namely Lemmas 3.1-3.3, follows from Lemmas 2-4 in Shen et al. (2014). So we state the results here without giving the proof.

**Lemma 5.3.1.** Define the discounted price of the investment fund as follows:

$$
\tilde{S}(t) := \exp \left\{ - \int_0^t r(s) \, ds \right\} S(t), \quad \tilde{S}(0) = S_0, \quad t \in T.
$$

According to Eq. (5.2.1) and Itô’s differentiation rule, we can get the dynamic of the discounted price process as follows

$$
\tilde{S}(t) = S_0 + \int_0^t \tilde{S}(s-) (\mu(s) - r(s)) \, ds + \int_0^t \tilde{S}(s-) \sigma(s) \, dW(s) + \int_0^t \tilde{S}(s-) \sum_{l=1}^N (e^{\beta_l(s-)} - 1) d\Phi_l(s) .
$$

Then the discounted price process $\tilde{S} := \{\tilde{S}(t) \mid t \in T\}$ is a $(\mathcal{G}, \mathcal{Q}^\theta)$-(local)-martingale if and only if the Esscher transform parameter $\theta$ satisfies the following equation:

$$
\mu(t) - r(t) + \theta(t) \sigma^2(t) + \sum_{l=1}^N (e^{\beta_l(t)} - 1) (e^{\beta_l(t)} - 1) a_l(t) = 0 . \tag{5.3.4}
$$

**Remark 5.3.1.** When $X(t) = e_j$, for each $j = 1, 2, \ldots, N$, Eq. (5.3.4) becomes

$$
\mu_j - r_j + \theta_j \sigma_j^2 + \sum_{l=1,l \neq j}^N (e^{\beta_{jl}(t)} - 1) (e^{\beta_{jl}(t)} - 1) a_{jl} = 0 . \tag{5.3.5}
$$

As discussed in Shen and Siu (2015b), the regime-switching Esscher transform parameter is uniquely determined when the model parameters are given. This means that a unique equivalent martingale measure can be determined via the Esscher transformation approach. This is why we say that the regime-switching risk is endogenously determined by the regime-switching Esscher transform.

**Lemma 5.3.2.** For each $t \in T$, let

$$
W^\theta(t) := W(t) - \int_0^t \theta(s) \sigma(s) \, ds ,
$$

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and
\[
\Phi^\theta_l(t) := \Phi_l(t) - \Phi_l(t) - \int_0^t a^\theta_l(s-)ds,
\]
where
\[
\Phi^\theta_l(t) := e^{\theta(t)\beta_l(t)}\Phi_l(t) = \sum_{j=1, j\neq l}^N e^{\theta_j\beta_l}a_{jl} \int_0^t \langle X(s-), e_j \rangle ds,
\]
and
\[
a^\theta_l(t) := e^{\theta(t)\beta_l(t)}a_l(t) = \sum_{j=1, j\neq l}^N e^{\theta_j\beta_l}a_{jl} \langle X(t), e_j \rangle.
\]

Then \( W^\theta := \{W^\theta(t)|t \in \mathcal{T}\} \) is a standard Brownian motion under \( Q^\theta \), and \( \Phi^\theta_l := \{\Phi^\theta_l(t)|t \in \mathcal{T}\} \) is an \((\mathbb{F}^X, Q^\theta)\)-martingale, for each \( l = 1, 2, \ldots, N \).

Furthermore, suppose \( A^\theta \) is an \((N \times N)\)-matrix with the following entries:
\[
a^\theta_{jl} := \begin{cases} 
e 1, j \neq l, \\
e 1, j = l. 
\end{cases}
\]

Then the chain \( X \) has the following semimartingale decomposition under \( Q^\theta \)
\[
X(t) = X(0) + \int_0^t A^\theta X(s)ds + M^\theta(t),
\]
where \( M^\theta := \{M^\theta(t)|t \in \mathcal{T}\} \) is an \( \mathbb{R}^N \)-valued, \((\mathbb{F}^X, Q^\theta)\)-martingale.

**Lemma 5.3.3.** Under \( Q^\theta \), the return process are given by:
\[
dY(t) = \left[r(t-) - \frac{1}{2} \sigma^2(t-) - \sum_{i=1}^N e^{\theta(t-)\beta_i(t-)}(e^{\beta_i(t-)} - 1 - \beta_i(t-))a_i(t-)\right]dt
\]
\[ + \sigma(t-)dW^\theta(t) + \sum_{l=1}^{N} \beta_l(t-)d\tilde{\Phi}_l(t) , \quad t \in T . \tag{5.3.6} \]

### 5.3.2 Minimal martingale measure

In this subsection, we consider how to determine the minimal martingale measure following Föllmer and Schweizer (1991). Being a semimartingale, the discounted price process \( \{e^S(t)|t \in T\} \) can be written as

\[
\tilde{S}(t) = S_0 + \Gamma(t) + \Upsilon(t) ,
\]

where

\[
\Gamma(t) = \int_{0}^{t} \tilde{S}(s-)(\mu(s) - r(s))ds \]
\[
\Upsilon(t) = \int_{0}^{t} \tilde{S}(s-){\sigma(s)dW(s)} + \int_{0}^{t} \tilde{S}(s-)[e^{\beta(s-)} - 1) d\tilde{\Phi}_l(s) .
\]

The following definition of the minimal martingale measure was given in Föllmer and Schweizer (1991).

**Definition 5.3.1.** An equivalent martingale measure \( Q^m \) with respect to \( \mathcal{P} \) is called the minimal martingale measure if the following conditions hold:

1. \( Q^m = \mathcal{P} \) on \( \mathcal{F}_0 \);

2. Define \( L \) as a square-integrable \( \mathcal{P} \)-martingale, orthogonal to \( \Upsilon \) under the original measure. After the measure change, \( L \) remains a martingale under \( Q^m \).

Theorem 5.3.1 presents the expression of the minimal martingale measure.

**Theorem 5.3.1.** Define a new probability measure \( Q^m \) equivalent to \( \mathcal{P} \) on \( \mathcal{G}(T) \) by the following Radon-Nikodym derivative:

\[
\frac{dQ^m}{d\mathcal{P}} \bigg|_{\mathcal{G}(T)} := Z(T) ,
\]

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where
\[
Z(t) = \exp \left\{ \int_0^t \frac{-(\mu(s) - r(s))\sigma(s)}{\sigma^2(s) + \sum_{l=1}^N (e^{\beta_l(s)} - 1)^2 a_l(s)} dW(s) - \frac{1}{2} \int_0^t \frac{(\mu(s) - r(s))^2 \sigma^2(s)}{\sigma^2(s) + \sum_{l=1}^N (e^{\beta_l(s)} - 1)^2 a_l(s))^2} ds \right. \\
+ \int_0^t \sum_{l=1}^N \frac{(\mu(s) - r(s))(e^{\beta_l(s)} - 1)}{\sigma^2(s) + \sum_{l=1}^N (e^{\beta_l(s)} - 1)^2 a_l(s)} a_l(s) ds \\
+ \int_0^t \sum_{l=1}^N \ln \left( 1 - \frac{(\mu(s) - r(s))(e^{\beta_l(s)} - 1)}{\sigma^2(s) + \sum_{l=1}^N (e^{\beta_l(s)} - 1)^2 a_l(s)} \right) d\Phi_l(s) \right\}. \tag{5.3.7}
\]

Furthermore, under the following assumptions,
1. \( E \left[ \exp \left\{ \int_0^t \frac{(\mu(s) - r(s))^2 ds}{\sigma^2(s)} + \int_0^t \sum_{l=1}^N \frac{(\mu(s) - r(s))^2 (e^{\beta_l(s)} - 1)^2 ds}{\sigma^4(s)} \right\} \right] < \infty, \quad \text{for} \quad t \in \mathcal{T}. \)
2. \( \frac{(\mu(t) - r(t))(e^{\beta_l(t)} - 1)}{\sigma^2(t) + \sum_{l=1}^N (e^{\beta_l(t)} - 1)^2 a_l(t)} < 1, \quad \text{a.s. for} \quad t \in \mathcal{T}. \)

then \( Q^m \) is the unique minimal martingale measure.

**Proof.** The proof of Theorem 5.3.1 resembles to Theorem 1 in Föllmer and Schweizer (1991) and Theorem 3.1 in Su et al. (2012). Here we only present the key steps of the proof.

- **Uniqueness.**

Suppose \( \mathcal{Q} \) is a minimal martingale measure. Define \( \{ \hat{Z}(t) | t \in \mathcal{T} \} \) as
\[
\hat{Z}(t) = E \left[ \frac{d\mathcal{Q}}{d\mathcal{P}} \bigg| \mathcal{G}(t) \right].
\]

Then, according to Kunita-Watanabe decomposition,
\[
\hat{Z}(t) = \hat{Z}(0) + \int_0^t \vartheta(s)d\Upsilon(s) + L(t), \quad \hat{Z}(0) = 1.
\]

Here, \( L \) is a square-integrable martingale and orthogonal to \( \Upsilon \) under \( \mathcal{P} \) and \( \vartheta \) is a predictable process satisfying \( E \left[ \int_0^T \vartheta^2(s)d\langle \Upsilon, \Upsilon \rangle \right] < \infty. \) Using Girsanov’s
theorem (see, for example, Protter (2004)), it is easy to see that the predictable process of finite variation under the new probability measure $\hat{Q}$ can be written as

$$\Gamma(t) = -\int_0^t \frac{1}{Z(s-)} d\left<\Upsilon, \hat{Z}\right>(s).$$

According to the definition of the minimal martingale measure, $\langle L, L \rangle = \langle L, \hat{Z} \rangle = 0$. Hence $\hat{Z}(t) = 1 + \int_0^t \vartheta(s) d\Upsilon(s)$ and $d\Gamma(t) = -\frac{\vartheta(t)}{Z(t-)} d\left<\Upsilon, \Upsilon\right>(t)$. Then

$$\hat{Z}(t) = 1 - \int_0^t \hat{Z}(s-) \frac{d\Gamma(s)}{d\left<\Upsilon, \Upsilon\right>(s)} d\Upsilon(s). \quad (5.3.8)$$

Since there exists a unique solution of Eq. (5.3.8), we obtain $\hat{Z} = Z$ and the uniqueness of the minimal martingale measure if there exists a minimal martingale measure.

• Existence.

By the Doob-Meyer decomposition of $\tilde{S}$ and the Doléans-Dade exponential, we can solve the explicit form of $Z(t)$, which is given by Eq. (5.3.7). Note that $Z$ is a square-integrable martingale under $\mathcal{P}$ when the assumptions 1 and 2 hold. Consequently, if $Q^m$ exists, $Z(t)$ is a square-integrable martingale under the original measure. Conversely, if $Z(t)$ is given by Eq. (5.3.7), we intend to verify the martingale measure $Q^m$ is the minimal martingale measure. Define $L$ as a square-integrable $\mathcal{P}$-martingale orthogonal to $\Upsilon$. Since $Z$ is the solution of Eq. (5.3.8), $\langle L, Z \rangle = 0$, verifying that $L$ is a local martingale under $Q^m$. Note that $L$ is a square-integrable martingale. So $L$ is a martingale under $Q^m$ and $Q^m$ is the minimal martingale measure.
Similar to discussions in Subsection 5.3.1, we also present the (local)-martingale condition and the dynamics of the Brownian motion, the Markov chain and the log-arithmic return process under the minimal martingale measure. The derivations of Lemmas 5.3.4, 5.3.5 and 5.3.6 resemble to those of Lemmas 5.3.1, 5.3.2 and 5.3.3. So we state the results without giving the proofs.

**Lemma 5.3.4.** The (local)-martingale condition is given by

\[
\mu(t) - r(t) - \frac{(\mu(t) - r(t))\sigma^2(t)}{\sigma^2(t) + \sum_{l=1}^{N} (e^{\beta_l(t)} - 1)^2 a_l(t)} - \sum_{l=1}^{N} \frac{(\mu(t) - r(t))(e^{\beta_l(t)} - 1)^2}{\sigma^2(t) + \sum_{l=1}^{N} (e^{\beta_l(t)} - 1)^2 a_l(t)} a_l(t) = 0 \tag{5.3.9}
\]

**Remark 5.3.2.** It is not difficult to see the martingale condition (5.3.9) always holds no matter what values \(\mu(t), r(t), \sigma(t)\) and \(\beta_l(t)\) take, for each \(l = 1, 2, \ldots, N\) and \(t \in \mathcal{T}\).

**Lemma 5.3.5.** For each \(t \in \mathcal{T}\), let

\[
W^m(t) := W(t) + \int_{0}^{t} \frac{(\mu(s) - r(s))\sigma(s)}{\sigma^2(s) + \sum_{l=1}^{N} (e^{\beta_l(s)} - 1)^2 a_l(s)} ds ,
\]

and

\[
\Phi^m_l(t) := \Phi_l(t) - \phi^m_l(t) = \Phi_l(t) - \int_{0}^{t} a^m_l(s-) ds ,
\]

where

\[
a^m_l(t) := \left(1 - \frac{(\mu(t) - r(t))(e^{\beta_l(t)} - 1)}{\sigma^2(t) + \sum_{l=1}^{N} (e^{\beta_l(t)} - 1)^2 a_l(t)} \right) a_l(t) .
\]

Then \(W^m := \{W^m(t) | t \in \mathcal{T}\}\) is a standard Brownian motion under \(Q^m\), and \(\Phi^m_l := \{\Phi^m_l(t) | t \in \mathcal{T}\}\) is an \((\mathbb{P}^X, Q^m)\)-martingale, for each \(l = 1, 2, \ldots, N\).
Lemma 5.3.6. Under $Q^n$, the return process is given by:

$$dY(t) = \left[ r(t-) - \frac{1}{2} \sigma^2(t-) - \sum_{l=1}^{N} (e^{\beta_l(t-)} - 1 - \beta_l(t-)) a^n_l(t-) \right] dt$$

$$+ \sigma(t-) dW^n(t) + \sum_{l=1}^{N} \beta_l(t-) d\Phi^n_l(t), \quad t \in T. \quad (5.3.10)$$

Remark 5.3.3. If we assume the jump ratio in the investment fund price level when the chain transits from state $e_j$ to state $e_l$ becomes zero, the minimal martingale measure $Q^n$ will be given by

$$dQ^n dP \bigg|_{G(T)} = \exp \left\{ \int_{0}^{T} \left( \frac{\mu(t) - r(t)}{\sigma(t)} \right) dW(t) - \frac{1}{2} \int_{0}^{T} \left( \frac{\mu(t) - r(t)}{\sigma(t)} \right)^2 dt \right\}.$$  

On the other hand, with this assumption, the Radon-Nikodym derivative of the regime-switching Esscher transform becomes

$$dQ^\theta dP \bigg|_{G(T)} = \exp \left\{ \int_{0}^{T} \theta(t) \sigma(t) dW(t) - \frac{1}{2} \int_{0}^{T} \theta^2(t) \sigma^2(t) dt \right\}.$$ 

According to the martingale condition, the regime-switching Esscher transform parameter can be determined uniquely by

$$\tilde{\theta}(t) = \frac{r(t) - \mu(t)}{\sigma^2(t)}. $$

Consequently, in this case, the Esscher transformed equivalent martingale measure is consistent with the minimal martingale measure. This is the same as the regime-switching Esscher transform in Elliott et al. (2005). This result is also illustrated in our numerical examples.

Remark 5.3.4. Compared with Yuen and Yang (2009), the main advantage of both the regime-switching Esscher transform and the minimal martingale measure method is that these two approaches can determine a unique pricing kernel, while Yuen and
Yang (2009) cannot select a unique pricing kernel. The derivation of the local-risk-minimizing measure under a Markov regime-switching jump-diffusion model, where the jump component is relating to the Markov chain, was considered in Elliott et al. (2011b).

5.4 Valuation of equity-indexed annuities

In this section, we will discuss the valuation of equity-indexed annuities with guarantees, including the point-to-point EIAs and the annual ratchet EIAs, under a double regime-switching model. For the point-to-point EIAs, analytical pricing formulae are derived by the inverse Fourier transform and the FFT method is applied to calculate the prices of the point-to-point EIAs. For the annual ratchet EIAs, the valuation relies on numerical methods. Although the derivations here involve some standard mathematical techniques, the analytical pricing formulae derived might hopefully be of some practical values. The proofs presented in this section are standard and do not involve new mathematical results. They are presented here for the ease of exposition.

For notational simplicity, we denote $Q^*$ as the risk-neutral martingale measure selected by either the regime-switching Esscher transform or the minimal martingale measure in Section 5.3. $E^*[\cdot]$ is the expectation under the risk-neutral probability measure $Q^*$. Under $Q^*$, the dynamics of the return process and the Markov chain are, respectively, given by:

$$dY(t) = \left[ r(t-) - \frac{1}{2}\sigma^2(t-) - \sum_{l=1}^{N} \left( e^{\beta_l(t-)} - 1 - \beta_l(t-) \right) a_l^*(t-) \right] dt$$

$$+ \sigma(t-)dW^*(t) + \sum_{l=1}^{N} \beta_l(t-)d\Phi_l^*(t), \quad t \in T,$$

(5.4.1)
and

\[ X(t) = X(0) + \int_0^t A^* X(s) ds + M^*(t), \quad t \in \mathcal{T}, \]

where \( W^* := \{ W^*(t) | t \in \mathcal{T} \} \) is a standard Brownian motion under \( Q^*; \Phi^*_t := \{ \Phi^*_t(t) | t \in \mathcal{T} \} \) is an \( (\mathbb{F}^X, Q^*) \)-martingale; \( A^* := [a^*_{jl}]_{j,l=1,2,...,N} \) is the rate matrix of the Markov chain under \( Q^*; M^* := \{ M^*(t) | t \in \mathcal{T} \} \) is an \( \mathbb{R}^N \)-valued, \( (\mathbb{F}^X, Q^*) \)-martingale.

### 5.4.1 Valuation of point-to-point EIAs

In this subsection we discuss the valuation of the point-to-point EIAs, the simplest type of EIAs. In year \( t \), the payoff of the point-to-point EIA \( G_{pp}(t) \) is given by

\[ G_{pp}(t) := \max\{\min(e^{\omega Y(t)}, e^{zt}), e^{zt}\}, \quad (5.4.2) \]

where \( \omega \) is the participation rate, \( z \) is the continuously compounded minimum guarantee rate and \( \gamma \) is the continuously compounded cap rate.

If the policyholder survives until the maturity of the contract, the policyholder will receive \( G_{pp}(T) \) at time \( T \). If the policyholder dies during the time period \( (t, t+1] \), for \( t = 1, 2, \ldots, T \), the beneficiary will receive \( G_{pp}(t) \) at \( t + 1 \). Consequently, the payoff of a point-to-point EIA with mortality risk is given by

\[
\begin{cases}
  G_{pp}(t), & t - 1 < T(x) \leq t, \quad \text{for} \quad t = 1, 2, \ldots, T, \\
  G_{pp}(T), & T(x) > T,
\end{cases}
\]

where \( T(x) \) is the future lifetime of a policyholder with purchased age \( x \). Suppose \( T(x) \) is independent of the price processes in the financial market. \( \tau p_x \) denotes the probability that the aged-\( x \) policyholder will survive until the contract expires at \( T \) and \( \tau q_x \) represents the probability that the aged-\( x \) policyholder dies in \( t \) years. The
mathematical definitions are given by \( TP_x = P(T(x) \geq T) \) and \( tQ_x = 1 - tP_x = 1 - P(T(x) \geq t) \).

Define the conditional characteristic function of \( Y(t) \) given \( F^X(t) \) under \( Q^* \) as:

\[
\varphi_{Y(t)|F^X(t)}(0, t, v) := E^*[e^{ivY(t)}|F^X(t)],
\]

and the unconditional discounted characteristic function of \( Y(t) \) under \( Q^* \) as:

\[
\tilde{\varphi}_{Y(t)}(0, t, v) := E^*\left[ e^{-\int_0^t r(s)ds} \varphi_{Y(t)|F^X(t)}(0, t, v) \right].
\]

The following lemma is standard. It resembles to Theorem 1 in Shen et al. (2014), (see also Proposition 3.1 in Fan et al. (2014) and Theorem 4.1 in Shen and Siu (2013a)).

**Lemma 5.4.1.** Under the risk-neutral probability measure \( Q^* \), the time-zero value of a European call option with maturity time \( t > 0 \) and payoff \( \max(e^{\omega Y(t)} - e^k) \) is given by

\[
C(0, t, k) := E^*\left[ \exp\left( -\int_0^t r(s)ds \right) (e^{\omega Y(t)} - e^k)_+ \right]
\]

\[
= \frac{e^{-\alpha k}}{\pi} \int_0^\infty e^{-ivk} \psi(0, t, v) dv ,
\]

where

\[
\psi(0, t, v) = \frac{\langle X(0) \exp\left[\left(\text{diag}(g(v - i(\alpha + 1))\omega) + B^*\right)t\right], 1\rangle}{\alpha^2 + \alpha - v^2 + (2\alpha + 1)iv} . \tag{5.4.3}
\]

with \( g(v) := (g_1(v), g_2(v), \ldots, g_N(v))^\prime \) and \( B^* = [b^*_{jl}]_{j,l=1,\ldots,N} \) given by

\[
g_j(v) := -r_j + iv(r_j - \frac{1}{2}\sigma_j^2) - \frac{1}{2}v^2\sigma_j^2
\]

\[
+ \sum_{l=1,l\neq j}^N \left( e^{iv\beta_{jl}} - 1 \right) - iv(e^{\beta_{jl}} - 1) d^*_{jl} , \tag{5.4.4}
\]

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and

\[
b_{jl}^* = \begin{cases} e^{i\nu_j}a_{jl}^* , & j \neq l , \\
- \sum_{l=1, l \neq j}^{N} e^{i\nu_j}a_{jl}^* , & j = l .
\end{cases}
\tag{5.4.5}
\]

**Proof.** The proof is standard. We include it here for the sake of completeness. Write

\[
R_t := \int_0^t r(s) ds.
\]

Let \( F_{Y(t)|F^X(t)}(y) \) be the conditional distribution function of \( Y(t) \) given \( F^X(t) \) under \( Q^* \). Following Carr and Madan (1999) and Liu et al. (2006), let \( \alpha \) be the dampening coefficient, we can calculate the Fourier transform of the dampened call option price as follows:

\[
\psi(0, t, v) := \int_{\mathbb{R}} e^{ivk} e^{\alpha k} C(0, t, k) dk
\]
\[
= \int_{\mathbb{R}} e^{ivk} e^{\alpha k} E^*[e^{-R_t (e^{\omega Y(t)} - e^k)}] dk
\]
\[
= E^* \left[ \int_{\mathbb{R}} e^{ivk} e^{\alpha k} \int_{\mathbb{R}} e^{\omega y} e^{(\alpha + iv)k} F_{Y(t)|F^X(t)}(dy) dk \right]
\]
\[
= E^* \left[ \int_{\mathbb{R}} e^{-R_t} \int_{-\infty}^{\omega y} e^{\omega y} e^{(\alpha + iv)k} - e^{(1+\alpha + iv)k} dk F_{Y(t)|F^X(t)}(dy) \right]
\]
\[
= E^* \left[ e^{-R_t} \tilde{\varphi}_{Y(t)|F^X(t)}\left( \frac{1}{\alpha + iv} - \frac{1}{1 + \alpha + iv} \right) \right]
\]
\[
= \frac{\tilde{\varphi}_{Y(t)}(0, t, (v - i(\alpha + 1))\omega)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v},
\tag{5.4.6}
\]

Therefore, we can derive the value of the Fourier transform of the option if we know the formulae of the unconditional, discounted characteristic function of \( Y(t) \) under \( Q^* \).

To make our paper self-contained, we briefly discuss the derivation. Detailed proof can be referred to Lemma 6 in Shen et al. (2014). Applying Itô’s differentiation rule to
\( e^{iuY(s)} \) gives

\[
dde^{iuY(s)} = e^{iuY(s)} \left\{ \left[ iu(r(s) - \frac{1}{2}\sigma^2(s)) - \frac{1}{2}u^2\sigma^2(s) + \sum_{l=1}^{N} \left( e^{ib_l(s)} - 1 \right) \right] ds + iu\sigma(s)dW^*(s) + \sum_{l=1}^{N} \left( e^{ib_l(s)} - 1 \right) d\Phi_l^*(s) \right\}.
\]

Then

\[
d\tilde{\varphi}_{Y(s)|X(s)}(0, s, v) = \tilde{\varphi}_{Y(s)|X(s)}(0, s, v) \left\{ \left[ -r(s) + iv(r(s) - \frac{1}{2}\sigma^2(s)) - v^2\sigma^2(s) \right] + \sum_{l=1}^{N} \left( e^{ib_l(s)} - 1 \right) a_l^*(s) \right\} ds + \sum_{l=1}^{N} \left( e^{ib_l(s)} - 1 \right) d\Phi_l^*(s) \right\}.
\]

Define

\[ h(s, v) := X(s)\tilde{\varphi}_{Y(s)|X(s)}(0, s, v), \quad s \in \mathcal{T}. \]

Using the stochastic integration by parts,

\[
dh(s, v) = (\text{diag}(g(v)) + A^*)h(s, v)ds + h(s, v)\sum_{l=1}^{N} \left( e^{ib_l(s)} - 1 \right) d\Phi_l^*(s) + \tilde{\varphi}_{Y(s)|X(s)}(0, s, v) dM^*(s) + \Delta X(s) \Delta \tilde{\varphi}_{Y(s)|X(s)}(0, s, v). \quad (5.4.7)
\]

Denote by

\[ A_0^* := A^* - \text{diag} \left[ (a_{11}^*, a_{22}^*, \ldots, a_{NN}^*)' \right], \]

\[ B_0^* := B^* - \text{diag} \left[ (b_{11}^*, b_{22}^*, \ldots, b_{NN}^*)' \right], \]

\[ D_0 := [d_{jl}]_{j,l=1,2,\ldots,N} - \text{diag} \left[ (d_{11}, d_{22}, \ldots, d_{NN})' \right], \]

with \( d_{jl} = b_{jl}^*/a_{jl}^* \), for each \( j, l = 1, 2, \ldots, N \). Note that

\[
\sum_{l=1}^{N} \left( e^{ib_l(s)} - 1 \right) d\Phi_l^*(s) = (D_0X(s) - 1 + X(s))'d\Phi^*(s),
\]

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\[ X(s) = X(0) + \int_0^s (I - X(u-1'))d\Phi^*(u) , \]

where \(1 := (1,1,\ldots,1) \in \mathbb{R}^N, \Phi^* := (\Phi^*_1, \Phi^*_2, \ldots, \Phi^*_N)' \in \mathbb{R}^N, \Phi^* := (\Phi^*_1, \Phi^*_2, \ldots, \Phi^*_N)' \in \mathbb{R}^N \) and \(I\) is an \((N \times N)\)-identity matrix. Then it is easy to check that

\[ \Delta X(s) \Delta \tilde{\varphi}_{Y(s)}|_{\mathcal{F}_X(s)}(0, s, v) \]
\[ = (I - X(s)1')\Delta \Phi^*(s)(D_0X(s) - 1 + X(s))' \Delta \Phi^*(s)\tilde{\varphi}_{Y(s)}|_{\mathcal{F}_X(s)}(0, s, v) \]
\[ = (I - X(s)1')\text{diag}(d\tilde{\Phi}^*(s))(D_0X(s) - 1)\tilde{\varphi}_{Y(s)}|_{\mathcal{F}_X(s)}(0, s, v) \]
\[ + (B^* - A^*)h(s, v)ds . \]

Then taking expectation on both sides of (5.4.7) under \(Q^*\) gives:

\[ dE^*[h(s, v)] = (\text{diag}(g(v)) + B^*)E^*[h(s, v)]ds . \]

Solving gives

\[ E^*[h(t, v)] = X(0) \exp [(\text{diag}(g(v)) + B^*)t] . \]

Consequently,

\[ \tilde{\varphi}_{Y(t)}(0, t, v) = \langle E^*[h(t, v)], 1 \rangle \]
\[ = \langle X(0) \exp [(\text{diag}(g(v)) + B^*)t], 1 \rangle \ . \]

Adopting the inverse Fourier transform, the analytical pricing formula for the European call option is obtained.

The proposition presents an analytical valuation formula for the EIA contract.

**Proposition 5.4.1.** Suppose an aged-\(x\) policyholder buys a \(T\)-maturity point-to-point EIA contract. Then the time-zero value of the EIA contract \(V_{pp}(0, T, \omega, \gamma, z)\) is given by

\[ V_{pp}(0, T, \omega, \gamma, z) \]
\[\sum_{t=1}^{T} E^*[e^{-\int_0^t r(s)ds} G_{pp}(t) 1_{t-1 < T(x) \leq t}] + E^*[e^{-\int_0^T r(s)ds} G_{pp}(T) 1_{T(x) > T}] \]

\[= \sum_{t=1}^{T} [I_{1}^{pp}(t) - I_{2}^{pp}(t) + I_{3}^{pp}(t)]_{t-1} p_x \cdot q_{x+t-1} + r p_x [I_{1}^{pp}(T) - I_{2}^{pp}(T) + I_{3}^{pp}(T)]_{T}, \]

where

\[I_{1}^{pp}(t) := E^*[e^{-\int_0^t r(s)ds} \max(e^{\omega Y(t)} - e^{zt}, 0)] \]  
\[= e^{-\alpha z t} \int_0^\infty e^{-ivzt} \psi(0, t, v)dv, \]  
\[I_{2}^{pp}(t) := E^*[e^{-\int_0^t r(s)ds} \max(e^{\omega Y(t)} - e^{\gamma t}, 0)] \]  
\[= e^{-\alpha \gamma t} \int_0^\infty e^{-iv\gamma t} \psi(0, t, v)dv, \]  
\[I_{3}^{pp}(t) := E^*[e^{-\int_0^t r(s)ds} e^{zt} \exp([A^* - \text{diag}(r')]t)], 1 \]  
\[= e^{zt} e^{-r_N t} (X(0) \exp([A^* - \text{diag}(r')]t)], 1 \]  

where \(r' := (r_1, r_2, \ldots, r_{N-1}, 0) \in \mathbb{R}^N.\)

Proof. Again the proof is standard. The payoff in Eq. (5.4.2) can be decomposed as:

\[G_{pp}(t) = \max(e^{\omega Y(t)} - e^{zt}, 0) - \max(e^{\omega Y(t)} - e^{\gamma t}, 0) + e^{zt}. \]  

Since \(T(x)\) is independent of \(Y(t)\) and \(X(t)\), its distribution will not change when we change the real-world probability measure to the risk-neutral one (please also refer to Proposition 3.2 in Lin et al. (2009)). Define \(1_A\) as the indicator function of the event \(A\). Under the assumption that financial and mortality risks are independent, the time-zero value of the EIA contract is given by

\[V_{pp}(0, T, \omega, \gamma, z) \]
\[= \sum_{t=1}^{T} E^*[e^{-\int_0^t r(s)ds} G_{pp}(t) 1_{t-1 < T(x) \leq t}] + E^*[e^{-\int_0^T r(s)ds} G_{pp}(T) 1_{T(x) > T}] \]
Furthermore, it is easy to see that

$$I_3^{pp}(t) = E^*\left[e^{-\int_0^t r(s)ds}e^{zt}\right]$$

$$= e^{zt}E^*\left[e^{-\int_0^t r(s)ds}\right]$$

$$= e^{zt}E^*\left[E^*\left[e^{-\int_0^t r(s)ds}|\mathcal{F}^X(t)\right]\right]$$

$$= e^{zt}e^{-r_N t} \langle X(0) \exp[(A^* - \text{diag}(r'))t], 1 \rangle ,$$

where \(r' := (r_1, r_2, \ldots, r_{N-1}, 0) \in \mathbb{R}^N\). The last equality follows Buffington and Elliott (2002).
Typically, the implied critical participation rate, which is calculated by solving the equation letting the EIA price be equal to 1, is generally reported in earlier works, see, for example, Lin et al. (2009). The following equation illustrates how to calculate these critical participation rates by solving the formula letting Eq. (5.4.8) be equal to 1, i.e.,

\[ V_{pp}(0, T, \omega^*, \gamma, z) = 1. \]

### 5.4.2 Valuation of the annual ratchet EIAs

In this subsection, the valuation of the annual ratchet EIAs is considered. The main feature of this kind of products is that the payoff functions are reset annually. In year \( t \), the contingent claims \( G_{ar}(t) \) is given by

\[ G_{ar}(t) := \prod_{i=1}^{t} \max\{\min(e^{\omega_{ar}(Y(i)-Y(i-1))}, e^{\gamma_{ar}}), e^{z_{ar}}\}, \tag{5.4.13} \]

where \( \omega_{ar} \) is the participation rate, \( z_{ar} \) is the minimum guarantee rate and \( \gamma_{ar} \) is the cap rate for the annual ratchet EIA. Similarly, if the policyholder survives until the maturity of the contract, the policyholder will receive \( G_{ar}(T) \) at time \( T \). If the policyholder dies during the time period \( (t, t+1], \) for \( t = 1, 2, \ldots, T \), the beneficiary will receive \( G_{ar}(t) \) at \( t+1 \). Consequently, the payoff of the annual ratchet EIA with mortality risk is given by

\[
\begin{cases}
G_{ar}(t), & t - 1 < T(x) \leq t, \quad \text{for} \quad t = 1, 2, \ldots, T, \\
G_{ar}(T), & T(x) > T,
\end{cases}
\]

The following result is obvious, so we just state the result.
Proposition 5.4.2. Suppose an aged-\(x\) policyholder buys the \(T\)-maturity annual ratchet EIA contract. Then under the risk-neutral probability measure \(Q^*\), the time-zero value of the EIA contract \(V_{ar}(0, T, \omega_{ar}, \gamma_{ar}, z_{ar})\) is given by

\[
V_{ar}(0, T, \omega_{ar}, \gamma_{ar}, z_{ar}) = \sum_{t=1}^{T} E^*[e^{-\int_0^t r(s)ds} G_{ar}(t)]_{t-1} p_x \cdot q_{x+t-1} + T p_x E^*[e^{-\int_0^T r(t)dt} G_{ar}(T)] \tag{5.4.14}
\]

and the implied critical participation rate can be calculated by solving

\[
V_{ar}(0, T, \omega_{ar}^*, \gamma_{ar}, z_{ar}) = 1.
\]

As pointed out in Lin et al. (2009), since it is impossible to know the state of the modulating Markov chain after the initiation time of the contract, it is difficult, if not impossible, to derive the analytical pricing formulae for the annual ratchet EIAs. Indeed, the return rates \(Y(t) - Y(t - 1)\) within each year, for \(t = 1, 2, \ldots, T\), are not independent under the regime-switching model. Instead, they are independent only given \(\mathcal{F}_X(T)\). So we can not use the independence property to calculate the expectation of a discounted product payoff function in Eq. (5.4.14) as the product of the expectations of a series of independent processes. Consequently, the FFT method can be hardly applied to calculate the prices of the annual ratchet EIAs. In the numerical examples, we illustrate the valuation of the annual ratchet EIAs under a double regime-switching model by using Monte Carlo methods. In Lin et al. (2009), closed-form pricing formulae are obtained for some designs of EIAs in the case of a two-state modulating Markov chain and the absence of jumps in the asset price triggered by state transitions. Here we use a different approach based on the FFT to derive analytical pricing formulae for some designs of EIAs for an \(N\)-state modulating Markov chain and the presence of jumps in the asset price triggered by state transitions.
5.5 Valuation of variable annuities

In this section, we investigate the valuation of a variable annuity with GMDB under the double regime-switching model.

Note that the cost of GMDB in VAs should be considered in the valuation process. Here, we also suppose no management fee or other expenses are charged and $\delta$ denotes the continuously compounded guarantee charge. Then, the after-the-charge logarithmic return $\hat{Y}(t)$ is governed by the following dynamics under the risk-neutral probability measure $Q^*$.

$$d\hat{Y}(t) = dY(t) - \delta dt$$
$$= \left[ r(t-) - \delta - \frac{1}{2}\sigma^2(t-) - \sum_{l=1}^{N} e^{\theta(t-)\beta_l(t-)}(e^{\beta_l(t-)} - 1 - \beta_l(t-))a_l(t-) \right] dt$$
$$+ \sigma(t-)dW^*(t) + \sum_{l=1}^{N} \beta_l(t-)d\Phi^*_l(t), \quad t \in \mathcal{T}.$$  \hspace{1cm} (5.5.1)

So the after-the-charge price of the investment fund satisfies $\hat{S}(t) := e^{-\delta t}S(t)$. In year $t$, the payoff of the variable annuity $G_{DB}(t)$ is given by

$$G_{DB}(t) := \max\{\hat{S}(t), e^{\rho t}\},$$

where $\rho$ is the continuously compounded minimum guarantee rate. Under this kind of contract, the larger one of the investment fund value and the death benefit will be paid to the beneficiary when the policyholder dies before the maturity time. If the policyholder survives until the contract expires, the value of the investment fund will be paid. Consequently, the payoff of a variable annuity with GMDB is given by

$$\begin{cases} 
G_{DB}(t), & t - 1 < T(x) \leq t, \quad \text{for} \quad t = 1, 2, \ldots, T, \\
\hat{S}(T), & T(x) > T.
\end{cases}$$
To simplify the notation, write $I_{DB}^1(t) = E^*[e^{-\int_0^t r(s)ds} \max (e^{Y(t)} - e^{\rho t}, 0)]$, $I_{DB}^2(t) = E^*[e^{-\int_0^t r(s)ds} e^{\rho t}]$ and $I_{DB}^3(T) = E^*[e^{-\int_0^T r(s)ds} \hat{S}(T)]$.

The following proposition is similar to Proposition 5.4.1.

**Proposition 5.5.1.** Suppose an aged-$x$ policyholder buys a $T$-maturity variable annuity with GMDB. Then the value of the GMDB in a variable annuity at time zero is given by

$$V_{GMDB}(0, T, \delta, \rho) = \sum_{t=1}^{T} E^*[e^{-\int_0^t r(s)ds} G_{DB}(t)_{t-1}p_x \cdot q_{x+t-1} + E^*[e^{-\int_0^T r(t)dt} \hat{S}(T)]_T p_x$$

$$= \sum_{t=1}^{T} (I_{DB}^1(t) + I_{DB}^2(t))_{t-1}p_x \cdot q_{x+t-1} + I_{DB}^3(T) T p_x , \quad \text{(5.5.2)}$$

where

$$I_{DB}^1(t) = E^*[e^{-\int_0^t r(s)ds} \max (e^{Y(t)} - e^{\rho t}, 0)] = \frac{e^{-\alpha \rho t}}{\pi} \int_{0}^{\infty} e^{-ivt} \psi_{DB}^t(0, t, v) dv ,$$

$$\psi_{DB}^t(0, t, v) = \exp(-(iv + (\alpha + 1))\delta t) \langle X(0) \exp \left[ (\text{diag}(g(v - i(\alpha + 1))) + B^*)t \right], 1 \rangle \frac{\alpha^2 + \alpha - v^2 + (2\alpha + 1)iv}{\alpha^2 + \alpha - v^2 + (2\alpha + 1)iv} .$$

Furthermore,

$$I_{DB}^2(t) = e^{\rho t} e^{-\rho T} \langle X(0) \exp[(A^* - \text{diag}(r'))t], 1 \rangle ,$$

where $r' := (r_1, r_2, \ldots, r_{N-1}, 0) \in \mathbb{R}^N$.

The proof is similar to Proposition 5.4.1, so we omit it.

**Remark 5.5.1.** Hedging of financial products are standard tools to manage financial risks. For equity-linked annuities, hedging is also of great importance. In practice,
in-house hedging programs have attracted more and more attention to manage the risks brought by equity-linked annuities. As discussed earlier, equity-linked annuities typically provide investors with minimum guaranteed benefits. Delta hedging and Rho hedging are two standard dynamic hedging methods. However, dynamic hedging also has some disadvantages, including the complex implementation, high transaction costs, etc. There also exist other ways to hedge the equity-linked annuities, such as (locally) risk-minimizing hedging, quantile hedging, etc. Lin and Tan (2003) considered the risk-minimizing hedging strategies for equity-indexed annuities under stochastic interest rate model. Some of the existing literature considered risk-minimizing hedging for options under regime-switching models (see, for example, Elliott and Osakwe, 2006; Deshpande and Ghosh, 2008; Basak et al., 2011; Elliott et al., 2011b; Qian et al., 2011; Su et al., 2012; etc.). Elliott and Osakwe (2006) and Coleman et al. (2007) investigated performances of delta hedging and (locally) risk-minimizing hedging and concluded that locally risk-minimizing hedging performs better than delta hedging. Examples of quantile hedging for equity-linked annuities under regime-switching models include Jin et al. (2011), Wang and Yin (2012). Consequently, evaluating performances of delta-hedging, risk-minimizing hedging and quantile hedging for equity-linked annuities under regime-switching models might be of great practical value.

5.6 Numerical examples

In this section, we perform numerical analysis for the valuation of the point-to-point EIAs, the annual ratchet EIAs and the GMDB in VAs under the double regime-switching model. As stated earlier, we adopt both the generalized regime-switching Esscher transform and the minimal martingale measure method to select equivalent martingale measures. Then, the FFT method is used to discretize the integral pricing
formulae for the guarantees in the point-to-point EIAs and the GMDB in VAs. For the valuation of the annual ratchet EIA, Monte-Carlo simulation is applied under the double regime-switching model.

To simplify our computation, we consider a two-state Markov chain $X$, where State 1 and State 2 of the chain represent a ‘Bad’ economy and a ‘Good’ one, respectively. Write $X(t) = (1, 0)'$ and $X(t) = (0, 1)'$ for State 1 and State 2, respectively. The classification of an economy here is based on the economic condition. For example, a “Good” economic state may represent the situation that the economy is booming while a “Bad” economic state may represent the situation that the economy is in recession. In economics, different variables or indicators are used to measure the economic condition. Interest rate is a commonly used macroeconomic variable which describes the performance of an economy. When the economy is good (bad), interest rate is usually high (low), which is in accord with the high (low) inflation rate. Furthermore, the volatility of a stock is generally low when the economy is good, while high when the economy performs badly. These make intuitive sense and have been verified by numerous empirical works. Consequently, in our analysis, it is reasonable to define a ‘Bad’ economy as one with a low interest rate and a high volatility, while a ‘Good’ economy

\[\text{The key step of applying fast Fourier transform (FFT) method to option valuation is the derivation of the characteristic function of the risk-neutral density. However, the option pricing formulae might not be square integrable, leading to the non-existence of the Fourier transform of the option price. This is the reason why Carr and Madan (1999) introduced the definition of a modified call price with a dampened coefficient. Carr and Madan (1999) gave a sufficient condition to guarantee the square integrability of the modified option price. Approximating errors are also key factors when a numerical method is adopted. For instance, truncation errors will be incurred when a discrete sum is used to approximate the semi-infinite integration of the option price. Consequently, error control in the numerical approximation should also be considered. Interested readers can refer to Carr and Madan (1999), Lee (2004), Liu et al. (2006) and Kwok et al. (2012).}\]
as one with a high interest rate and a low volatility. In practice, how to classify the state of an economy based on some economic indicators may be colored by some degrees of subjectivity. In particular, the choices of what economic indicators to be used or how to draw a line to distinguish a “Good” economic state from a “Bad” one based on some economic indicators may depend on some subjective judgments of the users of the model. In what follows, we give configurations of the hypothetical parameters values. The rate matrix of the chain $X$ under $P$ is given by

$$
A = \begin{pmatrix}
-a & a \\
  a & -a
\end{pmatrix},
$$

where $a$ takes discrete values from $\{0, 0.1, 0.2, \ldots, 1\}$ in our paper. Consider the following vectors for the appreciation rate, the risk-free interest rate, and volatility, respectively:

$$
\mu = (0.08, 0.10)', \quad r = (0.04, 0.08)', \quad \sigma = (0.3, 0.1)'.
$$

For ease of comparison, we also provide the numerical results for the valuation of the two kinds of equity-linked products under the single regime-switching model, which is the model considered in Fan (2013). Note that life table has been widely adopted to calculate the mortality of an insured by both academic researchers and market practitioners. Here we use the 2000-2003 China Life Table for male (see http://www.circ.gov.cn/) to depict the mortality rates of the policyholders in our model. The notifications of launching variable annuities announced by China Insurance Regulatory Commission (CIRC) in May 2011 means that variable annuities products are introduced officially to Chinese insurance market. However, the development of these products are slow in the last two years. There are various reasons why the development of the variable annuities market in China is slow. Here we do not intend to focus on discussing
these reasons. However, we believe that the development of scientific approaches to value variable annuities may have potentially constructive effects on the development of variable annuities market. So we use a China Life Table to illustrate how the model presented here may be applied to variable annuities which may be related to those traded in the Chinese insurance market. For illustration, we use the life table for male since similar results can be derived based on the life table for female.

1. The double regime-switching (DRS) model

The jump ratio of the double regime-switching model is determined by the following matrix

$$\beta = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}.$$

2. The single regime-switching (SRS) model

When the jump ratio remains zero during a state transition of the chain, the jump component of the double regime-switching model is absent. That is

$$\beta = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

In other words, the double regime-switching model reduces to the single one.

5.6.1 Numerical illustrations for the point-to-point EIA

Table 5.6.1 presents the prices of point-to-point EIAs with different cap rates $\gamma$, different maturities $T$ and different ages of the policyholders $x$ under the DRS model and the SRS model with the pricing kernel selected by the regime-switching Esscher transform. Here, we assume $\beta = 0.1$ for the DRS model. For both the two models, suppose
$S_0 = 1$, $a = 0.5$, $\omega = 0.8$ and $z = 0.03$. In the table, the value of “Underestimation” refers to the percentages of underestimation of the EIA prices under the SRS model when compared with those under the DRS model. In our examples, we assume the number of discretization $M = 4096$ in the FFT approach. Note that the embedded option prices converge very quickly. Under the assumption of the above hypothetical parameter values, the regime-switching Esscher transform parameters for the ‘Bad’ and ‘Good’ states are $\theta_1 = -0.4204$ and $\theta_2 = -1.3256$, respectively. Then the risk-neutral intensities of the chain are given by $a_{12}^\theta = 0.4794$ and $a_{21}^\theta = 0.5709$. Furthermore, the market prices of the regime-switching risk can also be calculated as $e^{\theta_1 \beta} - 1 = -0.0411$ and $e^{-\theta_2 \beta} - 1 = 0.1417$. We can see that regime switches have a significant impact on the market price of jump risk.

As shown in Table 5.6.1, the following results should be highlighted:

1. With the same minimum guarantee rate, the cap rate and the participation rate, the EIA prices in State 1 are systematically higher than those in State 2 under both models. These make intuitive sense. State 1 (‘Bad’ economy) has a lower interest rate and a higher volatility compared with State 2 (‘Good’ economy). Consequently, it is reasonable that the EIA prices in State 1 are higher than the corresponding prices in State 2 partly due to the additional amount of risk premium required to compensate for a ‘Bad’ economic condition. Another possible explanation is that the lower interest rate and the higher volatility in State 1 could increase the probability that the guarantee is triggered, leading to a higher price in State 1.

2. Since the risk-free interest rate and the volatility of the investment fund value process, as well as the generator of the Markov chain are assumed to be the same under the DRS model and the SRS model, the DRS model apparently gives higher
Table 5.6.1: Prices of point-to-point EIAs under the DRS model and the SRS model with the Esscher transformed equivalent martingale measure

<table>
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<td>$\text{State 2}$</td>
<td>$\text{State 1}$</td>
<td>$\text{State 2}$</td>
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<td>$\text{State 2}$</td>
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<td>1.74%</td>
<td>2.21%</td>
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<td>8</td>
<td>1.0440</td>
<td>1.0149</td>
<td>1.0235</td>
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<td>1.96%</td>
<td>2.35%</td>
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<tr>
<td>10</td>
<td>1.0252</td>
<td>0.9978</td>
<td>1.0033</td>
<td>0.9732</td>
<td>2.14%</td>
<td>2.47%</td>
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<td>$\text{State 2}$</td>
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<td>1.0450</td>
<td>1.0147</td>
<td>1.0244</td>
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<td>2.33%</td>
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<td>1.0223</td>
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<td>6</td>
<td>1.0637</td>
<td>1.0296</td>
<td>1.0458</td>
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<tr>
<td>8</td>
<td>1.0477</td>
<td>1.0165</td>
<td>1.0278</td>
<td>0.9934</td>
<td>1.90%</td>
<td>2.27%</td>
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<td>0.9990</td>
<td>1.0065</td>
<td>0.9749</td>
<td>2.09%</td>
<td>2.41%</td>
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EIA prices due to additional jump risk induced by state transitions. (i.e. the EIA prices are underestimated under the SRS model). Although the underestimation of EIA prices are not considerably large, the jump risk should never be ignored.

3. It is worth noting that the underestimation percentages are higher in State 2 than those in State 1. When the age of the insured increase, the underestimation percentage will decrease.

4. The prices of point-to-point EIAs calculated with a cap rate are lower than those calculated without a cap rate ($\gamma = \infty$). The presence of a cap rate implies that the potential upside profit received by the policyholder is bounded. Consequently, it is not unreasonable that the prices of point-to-point EIAs will be lower when there is a cap rate.

5. The prices of point-to-point EIAs will increase when the policyholder’s age increases. One possible explanation is that the mortality risk of the policyholder will increase when the underwriting age of the policyholder increases. If the insurer faces a higher mortality risk, the prices of EIAs will be more likely to increase. Another possible explanation is that the guarantee is more likely to be triggered by death of a more senior policyholder, which may lead to higher values of EIAs.

6. The prices of point-to-point EIAs are higher with shorter maturities. The explanation for this phenomenon could be similar with the one for item 5, since the effect of a higher policyholder’s age is similar to that of a shorter maturity.

Table 5.6.2 presents the prices of point-to-point EIAs with different cap rates $\gamma$, different maturities $T$ and different ages of the policyholders $x$ under the DRS model and the SRS model with the minimal martingale measure. To compare the EIAs prices
based on these two pricing kernels, we adopt the same configurations of parameter values as those adopted in Table 5.6.1. The values of “Underestimation” are percentages of underestimation of the EIA prices under the SRS model when compared with those under the DRS model. Comparing the results in Table 5.6.1 and 5.6.2, it is easy to notice that the prices of point-to-point EIAs calculated under the SRS model with the two pricing kernels are exactly the same. This is in line with the explanations in Remark 5.3.3. The prices of point-to-point EIAs under the DRS model obtained by the two equivalent martingale measures are very close. Furthermore, the highlighted results in Table 5.6.1 can also explained the prices in Table 5.6.2. In what follows, unless otherwise stated, we always apply the Esscher transformed equivalent martingale measure to calculate the prices of the annuity guarantees.

Table 5.6.3 presents the critical participation rates for the point-to-point EIA under the DRS model and the SRS model, respectively. Here, we assume $\beta = 0.1$ for the DRS model and $S_0 = 1$, $T = 10$, $a = 0.5$ and $z = 0.03$ for both models. In the table, the numbers in parentheses under the critical participation rates represent the percentages of underestimation of the critical participation rates under the SRS model when compared with those under the DRS model. The results for the aged-55, aged-60 and aged-65 policyholders are provided as well. From the comparisons between Table 5.6.1 and Table 5.6.3, we notice that the results of the EIA prices and the critical participation rates display totally opposite trends. These make intuitive sense. The following simple example is to illustrate the idea. There are two insurers, Insurer A and Insurer B, selling EIAs. Insurer A, may experience a higher risk than Insurer B. The two insurers have the same assumptions except the participation rate and the price of the EIA. If the two insurers are required to have the same participation rates, Insurer A will definitely increase the prices of EIAs to manage the higher risk. If the prices of EIAs provided by the two insurers are required to be the same, Insurer A can
Table 5.6.2: Prices of point-to-point EIAs under the DRS model and the SRS model with the minimal martingale measure

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<td>SRS model State 1</td>
<td>State 2</td>
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<td>1.0211</td>
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<td>1.85%</td>
<td>2.14%</td>
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<td>0.9978</td>
<td>0.9685</td>
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<td>2.24%</td>
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<td>State 2</td>
<td>SRS model State 1</td>
<td>State 2</td>
<td>Underestimation State 1</td>
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<td>1.0046</td>
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<td>1.0124</td>
<td>1.0235</td>
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<td>10</td>
<td>1.0231</td>
<td>0.9950</td>
<td>1.0033</td>
<td>0.9732</td>
<td>1.94%</td>
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Table 5.6.3: Critical participation rates for the point-to-point EIAs under the DRS model and the SRS model with the Esscher transformed equivalent martingale measure

<table>
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<th>$\gamma = \infty$</th>
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</thead>
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<td>$\text{SRS model}$</td>
<td>$\text{DRS model}$</td>
</tr>
<tr>
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<td>State 2</td>
<td>State 1</td>
</tr>
<tr>
<td>55</td>
<td>0.7676</td>
<td>0.8098</td>
<td>0.8036</td>
</tr>
<tr>
<td></td>
<td>(0.00%)</td>
<td>(0.00%)</td>
<td>(-4.69%)</td>
</tr>
<tr>
<td>60</td>
<td>0.7638</td>
<td>0.8074</td>
<td>0.8000</td>
</tr>
<tr>
<td></td>
<td>(0.00%)</td>
<td>(0.00%)</td>
<td>(-4.74%)</td>
</tr>
<tr>
<td>65</td>
<td>0.7575</td>
<td>0.8036</td>
<td>0.7942</td>
</tr>
<tr>
<td></td>
<td>(0.00%)</td>
<td>(0.00%)</td>
<td>(-4.84%)</td>
</tr>
</tbody>
</table>
only decrease the participation rate to limit the potential upside profit the customers could gain. So it is not unreasonable that the results in Table 5.6.1 and Table 5.6.3 show opposite trend. Consequently, the explanations for Table 5.6.3 can be similarly presented as those for Table 5.6.1.

Furthermore, we also assume $x = 60$, $T = 10$, $\beta = 0.1$, $S_0 = 1$, $a = 0.5$ and $\gamma = 0.2$ to illustrate EIA prices under the DRS model with different levels of participate rate $\omega$ in both State 1 and State 2. From Figs. 5.6.1 and 5.6.2, the EIA prices increase with the value of the participation rate $\omega$ in both states. In addition to the explanations earlier, Figs. 5.6.1 and 5.6.2 also illustrate the interplay between the EIA price and the participation rate. When other assumptions remain the same, a higher participation rate implies that the policyholder could gain higher potential upside profit, leading to a higher price of the EIA contract.

![Figure 5.6.1: EIA prices corresponding to different $\omega$ with $z = 0.03$](image)

![Figure 5.6.2: EIA prices corresponding to different $\omega$ with $z = 0.04$](image)

As in Table 5.6.1, we have discussed the EIA prices obtained under the DRS model with cap rate and without cap rate, respectively. Here, we provide the sensitivity analysis for the EIA prices with respect to the cap rate $\gamma$. Under the DRS model, we assume $x = 60$, $T = 10$, $\beta = 0.1$, $S_0 = 1$, $a = 0.5$ and $\omega = 0.8$. Figs. 5.6.3
and 5.6.4 illustrate that the prices of point-to-point EIAs will increase when the cap rate $\gamma$ increases. This may be attributed to the higher potential upside profit the policyholder may gain. A larger cap rate implies that the policyholder could receive a higher potential upside profit, leading to a higher value of the EIA contract.

Under the DRS model, we assume $x = 60, T = 10, \beta = 0.1, S_0 = 1, \omega = 0.8$ and $\gamma = 0.2$. A sensitivity analysis for the EIA prices with respect to the rate of transition $a$ is presented. Note that the probability of the transitions between different states of the economy increases with $a$, which implies a more volatile economy. The regime-switching effect is degenerate when $a = 0$. From Figs. 5.6.5 and 5.6.6, we notice that the prices of the EIAs decrease with $a$ in State 1 while increase with $a$ in State 2. As explained earlier, the EIAs are more expensive in State 1 (bad state) than in State 2 (good state). From this aspect, the prices of EIAs should decrease in State 1. However, the higher transition intensity in State 1 will lead to a higher transition probability of the chain from State 1 to State 2, hence a higher frequency of triggered jumps. Seen from this aspect, the market price of the jump risk may increase when the transition
intensity $a$ increases. Consequently, we further calculate the market prices of the diffusion risk and the jump risk for different transition intensities $a$. The market prices of the diffusion risk are $0.4444, 0.4345, 0.4250, 0.4159, 0.4071, 0.3988$ when the values of $a$ are $0, 0.2, 0.4, 0.6, 0.8, 1$ respectively. Meanwhile, the market prices of the jump risk are $-0.0435, -0.0425, -0.0416, -0.0407, -0.0399, -0.0391$, respectively. Under our configurations of the hypothetical values of the model parameters, the market prices of the diffusion risk will decrease with $a$, while those of the jump risk will increase with $a$. Consequently, the trend of the EIA prices with increasing transition intensities is determined by the combined effects of both the diffusion risk and the jump risk. Seen from the numerical results, although the jump may occur more frequently when $a$ increases, the market prices of the diffusion risk is the dominant factor. This provides some evidence that why the EIA prices will decrease when the transition intensity increases in State 1. On the other hand, the EIA prices in State 2 will increase when $a$ increases. This makes intuitive sense. When $a = 0$, the transition probabilities of the chain between State 1 and State 2 are zero, implying the regime switching effect does not exist under this degenerate case. Consequently, the EIA prices are the maximal in
State 1 and the minimal in State 2 when $a = 0$.

Furthermore, from Figs. 5.6.1-5.6.6, it is worth noting that the EIA prices calculated with $z = 0.04$ are higher than those calculated with $z = 0.03$. This also makes intuitive sense since a higher cost of guarantee, implied by a larger minimum guarantee, will lead to a higher price.

### 5.6.2 Numerical illustrations for the annual ratchet EIAs

From the discussions in the numerical examples for the valuation of the point-to-point EIAs, we only present the implied critical participation rates for the annual ratchet EIAs, which is defined by the following equation:

$$V_{ar}(0, T, \omega^{*}_{ar}, \gamma_{ar}, z_{ar}) = 1.$$  

Table 5.6.4 presents the critical participation rate for the annual ratchet EIAs under the DRS model and the SRS model, respectively. The parameter configurations are assumed to be the same as those used in the valuation of the point-to-point EIAs, i.e., $\beta = 0.1$ for the DRS model and $S_0 = 1$, $T = 10$, $a = 0.5$ and $z_{ar} = 0.03$ for both models. The results for the aged-55, aged-60 and aged-65 policyholders are provided as well. In table 5.6.4, the numbers in parentheses under the critical participation rates represent the percentages of overestimation of the annual ratchet EIA participation rates under the SRS model when compared with those under the DRS model.

As indicated in Table 5.6.4, it is easy to notice similar explanations of the results for the annual ratchet EIA valuation can be given as those for the point-to-point EIA valuation. Comparing Table 5.6.4 with Table 5.6.3, the critical participation rates for the annual ratchet EIAs are considerably lower than those for the point-to-point EIAs, which imply that the ratchet EIAs are much more valuable than the point-to-point EIAs. In addition, the overestimating percentage of the annual ratchet EIA
Table 5.6.4: Critical participation rates for the annual ratchet EIAs under the DRS model and the SRS model with the Esscher transformed equivalent martingale measure

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<td>State 2</td>
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<tr>
<td>55</td>
<td>0.3806</td>
<td>0.4347</td>
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<tr>
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<td>(0.00%)</td>
<td>(0.00%)</td>
</tr>
<tr>
<td>60</td>
<td>0.3787</td>
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<tr>
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<td>(0.00%)</td>
<td>(0.00%)</td>
</tr>
<tr>
<td>65</td>
<td>0.3774</td>
<td>0.4295</td>
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<tr>
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<td>(0.00%)</td>
<td>(0.00%)</td>
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<td>0.3719</td>
<td>0.4250</td>
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<tr>
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<td>(0.00%)</td>
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<tr>
<td>60</td>
<td>0.3701</td>
<td>0.4236</td>
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<td>(0.00%)</td>
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<td>0.3683</td>
<td>0.4205</td>
</tr>
<tr>
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<td>(0.00%)</td>
<td>(0.00%)</td>
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</table>
participation rates under the SRS model when compared with those under the DRS model are much higher than those of the point-to-point EIA participation rates.

5.6.3 Numerical illustrations for the GMDB

Table 5.6.5 and Table 5.6.6 present the prices of the GMDB in variable annuities with different minimum guarantee rates, different ages and different maturities under the DRS model and the SRS model with the Esscher transformed equivalent martingale measure and the minimal martingale measure, respectively. Here, we assume $\beta = 0.1$ for the DRS model. For both models, suppose $S_0 = 1$, $a = 0.5$ and $\delta = 0.03$. In Table 5.6.5 and Table 5.6.6, the numbers of “Underestimation” represent the percentages of underestimation of the GMDB prices under the SRS model when compared with those under the DRS model.

From Table 5.6.5 and Table 5.6.6, we have the following observations.

1. With the same minimum guarantee rate and maturity time, the prices of the GMDB in VAs in State 1 are systematically higher than those in State 2 under both models. This makes economic sense since State 1 (State 2) is a ‘Bad’ economy (a ‘Good’ one).

2. The prices of the GMDB in VAs are higher under the DRS model than those under the SRS model with the same configuration of hypothetical parameters values.

3. When the minimum guarantee rate increases, the prices of the GMDB in VAs will increase. This makes intuitive sense. To provide a higher minimum guarantee rate, the insurers will pay more attention to make investment strategies, implying a higher cost. Consequently, the prices of GMDB in VAs will be higher than those
Table 5.6.5: Prices of the GMDB in VAs under the DRS model and the SRS model with the Esscher transformed equivalent martingale measure

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<td>0.19%</td>
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<td>0.8550</td>
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<td>0.8333</td>
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<td>0.62%</td>
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Table 5.6.6: Prices of the GMDB in VAs under the DRS model and the SRS model with the minimal martingale measure

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<td>0.8062</td>
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<table>
<thead>
<tr>
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<td>0.7748</td>
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<td>0.23%</td>
<td>0.27%</td>
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<td>0.8446</td>
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<td>0.38%</td>
<td>0.46%</td>
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<tr>
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<td>0.8418</td>
<td>0.8327</td>
<td>0.8370</td>
<td>0.8270</td>
<td>0.57%</td>
<td>0.68%</td>
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</tr>
</tbody>
</table>
with lower minimum guarantee rates.

4. The prices of the GMDB in VAs increase when the policyholder’s age increases.

5. Contracts of the GMDB in VAs are more valuable when the maturity time is shorter.

Note that possible explanations for the above results are similar as those for the valuation of the point-to-point EIAs. It is worth mentioning that Tables 5.6.1 and 5.6.5 show that the EIA prices and the GMDB prices under the SRS model are lower than those under the DRS model. Consequently compared to those prices in Lin et al. (2009), the prices obtained from the DRS model provide more conservative estimates for the GMDB prices to compensate for the additional jump risk triggered by state transitions.

With the same configurations of the parameter values, Table 5.6.7 presents the fair guarantee charges for the GMDB in VAs under the DRS model and the SRS model corresponding to the GMDB prices in Table 5.6.5. As indicated in Table 5.6.7, the fair guarantee charges in State 1 are systematically higher than those in State 2 under both models. These make intuitive sense. State 1 (‘Bad’ economy) has a lower interest rate and higher volatility compared with State 2 (‘Good’ economy). Consequently, the fair guarantee charges in State 1 will be higher than those in State 2 due to the higher cost to maintain the minimum guarantee. This is also one possible explanation for the results that the DRS model requires higher fair guarantee charge than the SRS model.

Under the DRS model, we assume $x = 60$, $T = 10$, $\beta = 0.1$, $S_0 = 1$ and $a = 0.5$. Figs. 5.6.7 and 5.6.8 present the sensitivity analysis for the GMDB prices with respect to $\delta$. A larger $\delta$ implies a higher cost to maintain the minimum guarantee, leading to a lower GMDB price. This gives an intuitive description of the relationship between the GMDB prices and the fair guarantee charges.
Table 5.6.7: Fair guarantee charge for the GMDB in VAs under the DRS model and the SRS model with the Esscher transformed equivalent martingale measure

<table>
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<th>$\rho = 0.02$</th>
<th>(\text{DRS model})</th>
<th>(\text{SRS model})</th>
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<td>(x)</td>
<td>(\text{State 1})</td>
<td>(\text{State 2})</td>
</tr>
<tr>
<td>55</td>
<td>0.140</td>
<td>0.100</td>
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<tr>
<td>60</td>
<td>0.250</td>
<td>0.190</td>
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<td>65</td>
<td>0.450</td>
<td>0.340</td>
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<table>
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<th>$\rho = 0.03$</th>
<th>(\text{DRS model})</th>
<th>(\text{SRS model})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td>(\text{State 1})</td>
<td>(\text{State 2})</td>
</tr>
<tr>
<td>55</td>
<td>0.160</td>
<td>0.120</td>
</tr>
<tr>
<td>60</td>
<td>0.290</td>
<td>0.220</td>
</tr>
<tr>
<td>65</td>
<td>0.530</td>
<td>0.410</td>
</tr>
</tbody>
</table>
Under the assumptions that $x = 60$, $T = 10$, $\beta = 0.1$, $S_0 = 1$ and $\delta = 0.03$, we also illustrate GMDB prices under the DRS model with different levels of $a$ in both State 1 and State 2. Seen from Figs. 5.6.9 and 5.6.10, the GMDB prices decrease with $a$ in State 1, while increase with $a$ in State 2. The explanations are the same with those for the valuation of the point-to-point EIAs.
5.7 Conclusions

We considered the valuation of the point-to-point EIAs, the annual ratchet EIAs and the GMDB in VAs under the double regime-switching model, which can endogenously determine the regime-switching risk. To select an equivalent martingale measure, we applied the generalized version of the regime-switching Esscher transform. This can quantify the regime-switching risk. Then under the risk-neutral probability measure selected by the generalized regime-switching Esscher transform, analytical pricing formulae for the two products were obtained using the inverse Fourier transform method. The Fast Fourier Transform (FFT) is adopted to discretize the analytical pricing formulae. Numerical examples illustrated the regime-switching effects on the prices of the point-to-point EIAs, the annual ratchet EIAs and GMDB in VAs. Furthermore, our numerical examples could illustrate the impacts of different models and parameters on the prices of the equity-linked products. As given in Table 5.6.1 and Table 5.6.5, the SRS model adopted in Lin et al. (2009) might underestimate the prices of equity-linked products without incorporating the regime-switching risk. By incorporating those jumps triggered by state transitions, we can endogenously determine the regime-switching risk. The sensitivity analysis of different parameters also provides some intuition about the qualitative price comparisons.

Since the terms of the EIAs and other VA contracts are as long as several decades, it is natural to consider the uncertainty of mortality rate in such a long period. Consequently, it may be interesting to incorporate stochastic mortality in the current modeling framework. Another potential research direction is the derivation of analytical pricing formulae for VA contracts with some complicated optionality structures under the double regime-switching model. It seems that the Laplace transform may provide some clues to this possibly challenging problem.
Chapter 6

Pricing dynamic fund protection under hidden Markov models

6.1 Introduction

Compared with upside profits, investors often place greater emphasis on downside risk. Partly due to this reason, academic researchers and industry practitioners have innovated a variety of financial and insurance products to deal with the downside risk, including traditional put options. However, put options can only provide investors with protection at the exercise moment. Although many innovative investment protection plans are introduced to overcome this disadvantage, most of them are too sophisticated for investors to understand. Consequently the demand of effective and convenient investment plans is significant.

Among different investment protection plans, dynamic fund protections (DFP) have been one of the most popular extensions. Compared with traditional put options, DFP plans protect investors during the whole investment period. This feature is the
The main reason why the dynamic fund protection plans have attracted both academic researchers and investors’ attention. There are many forms of dynamic fund protection plans. The most fundamental one is that the investment fund value will be upgraded when the fund value falls below a certain protection level. This kind of protection plan was pioneered by Gerber and Shiu (1998). Since then, much attention has been paid to the DFP valuation when the value process of the underlying investment fund is governed by different stochastic models, such as the classical Black-Scholes model and the constant elasticity of variance (CEV) process (see Gerber and Shiu, 1999, 2003a, 2003b; Gerber and Pafumi, 2000; Imai and Boyle, 2001; Fung and Li, 2003; Chu and Kwok, 2004). Indeed, dynamic fund protection plans are products with lookback option features, which were shown in Imai and Boyle (2001), Chu and Kwok (2004) and Wong and Chan (2007). Wong and Chan (2007) considered the valuation of DFPs under a multiscale stochastic volatility model and derived semi-analytical formulas through asymptotic techniques. They also established the relationship between a fixed strike lookback call option and a DFP. In their paper, the parity equation between a fixed-strike lookback call option and a floating-strike lookback put option, derived in Wong and Kwok (2003), was used. Apart from pricing the DFPs, Tse et al. (2008) investigated the hedging of discrete-DFPs. Wong and Lam (2010) assumed that the investment fund follows a Lévy process and solved the valuation problem.

Regime-switching models are one of the major classes of models in financial econometrics. This class of model can incorporate the impacts of changes in economic conditions on financial and economic dynamics. Hamilton (1989) popularized the use of regime-switching models in financial econometrics. In the new millennium, there has been a considerable interest in applying regime-switching models to option valuation. Buffington and Elliott (2002) studied the valuation of European options and American options. Since the Markov chain will bring another source of uncertainty, the finan-
cial market is typically incomplete under the assumption of regime-switching models. Consequently, there is more than one equivalent martingale measure for valuation. Gerber and Shiu (1994) introduced the use of the Esscher transform to price options. The Esscher transform has been used to select an equivalent martingale measure in a regime-switching market (see Elliott et al., 2005; Siu, 2008; Boyle and Draviam, 2007; Yuen and Yang, 2010a; Siu et al. 2011; etc.).

It appears that most of the previous works on option valuation under regime-switching models suppose that the Markov chain is completely observable while, in practice, the states of the chain (or the states of the economy) may not be observed. Hidden Markov models present a natural choice for modelling transitions in hidden states of an economy. Assuming that the Markov chain is not observable, in a discrete-time regime-switching model, Ishijima and Kihara (2005) studied an option valuation problem using the locally risk-neutral valuation relationship. A higher-order hidden Markov model (HHMM) was considered in Ching et al. (2007). They assumed the dynamics of the asset price follow the HHMM and investigated the valuation of exotic options, including Asian options, barrier options and lookback options. Liew and Siu (2010) considered option valuation under a discrete-time hidden Markov regime-switching Gaussian model. They adopted both the Esscher transform and the extended Girsanov’s principle to select a pricing kernel.

It is known that some assumptions of the classical Black-Scholes model cannot capture certain vital features of financial time series. Considering uncertainties in business climates in the current market, structural changes in economic conditions are inevitable. These changes in economic conditions represent an additional source of risk to which an additional amount of risk premium may be required to compensate. How can one determine the additional amount of risk premium? The answer rests on how the regime-switching is priced. Consequently, the regime-switching risk should never be
ignored. Subscribing to this point of view, the prices of DFPs may be underestimated under the Black-Scholes model. To incorporate the impacts of the structural changes in economic conditions, there is a practical value to investigate the valuation of DFPs under a hidden Markov model. Note that our numerical examples also illustrates this statement. On the other hand, we use the Baum-Welch algorithm and the Viterbi algorithm to obtain the most-probable path for the hidden Markov chain. Then, maximum likelihood estimates of the model parameters are derived. The two-stage estimation method is intuitively appeal and easy to implement in practice. This approach may be more acceptable to market practitioners.

In this paper, we discuss how to price the dynamic fund protection plans when the dynamics of the investment fund are governed by a geometric Brownian motion whose coefficients are modulated by a hidden Markov chain. We assume that the values of the appreciation rate and the volatility will change when the states of the underlying continuous-time, finite-state hidden Markov chain switch. The states of the chain represent hidden states of an economy. The Esscher transform is employed to select a pricing kernel for the valuation of the DFP. Considering the lookback option feature in the DFP, we adopt a partial differential equation approach to price the floating strike lookback put options. Then, the prices of the corresponding fixed strike lookback call options can be calculated according to the put-call parity derived in Wong and Kwok (2003). To estimate the parameters of the hidden Markov model, we adopt the Baum-Welch algorithm, based on the discretization of continuous-time smoother. Then, Viterbi algorithm is applied to get the most-probable path of the Markov chain. We derive the maximum likelihood estimates of model parameters. To illustrate the valuation of DFP, we give numerical examples based on real financial data.

The rest of the paper is structured as follows: Section 6.2 presents the hidden Markov-modulated investment fund model. In section 6.3, we select a pricing kernel by
the Esscher transform. Section 6.4 presents the partial differential equation approach to value the dynamic fund protection. In the following section, we discuss how to estimate the states and transition matrix of the hidden Markov model. The maximum likelihood estimates of the model parameters are also derived. Some numerical examples are given in section 6.6. The final section summarizes the paper.

6.2 The model dynamics

In this section, we consider a continuous-time economy with two investment securities, namely, a risk-free bond $B$ and an investment fund $S$, which are continuously tradable on the time horizon $\mathcal{T}$. Here, $\mathcal{T} := [0, T]$, where $T < \infty$. We assume that there are no transaction costs and taxes involved in trading. Furthermore, all dividends are reinvested in the fund and any fractional units of the index can be traded. We present a hidden Markov regime-switching model for the underlying fund price. Suppose $(\Omega, \mathcal{F}, \mathcal{P})$ is a complete probability space, where $\mathcal{P}$ is a real-world probability measure. We adopt bold-face letters to denote matrices (or vectors), and write $y'$ for the transpose of a matrix, or a vector $y$. Let $1_E$ denote the indicator function of an event $E$.

We assume that the hidden states of an economy are modeled by a continuous-time, finite-state Markov chain $X := \{ X_t \mid t \in \mathcal{T} \}$. The Markov chain is defined on $(\Omega, \mathcal{F}, \mathcal{P})$ with the finite state space $S := \{ s_1, s_2, \ldots, s_N \}$. Without loss of generality, following Elliott et al. (1994), we can identify the state space of $X$ to be a finite set of standard unit vectors $\mathcal{E} := \{ e_1, e_2, \ldots, e_N \}$, where $e_i \in \mathbb{R}^N$ are column vectors and the $j$-th component of $e_i$ is the Kronecker delta $\delta_{ij}$, for each $i, j = 1, 2, \ldots, N$. We call $\mathcal{E}$ the canonical state space of the chain $X$. We define $A := [a_{ij}]_{i,j=1,2,\ldots,N}$ as the generator of the chain $X$ under $\mathcal{P}$, where $a_{ii} = -\sum_{j=0,j\neq i}^N a_{ij}, i = 0, 1, \ldots, N$. Note that the
transition probability, from time $t$ to $t + \Delta t$, is $I + A\Delta t$, where $I$ denotes the identity matrix.  

Elliott et al. (1994) obtained the following semi-martingale dynamics for the chain:

$$X_t = X_0 + \int_0^t AX_s ds + M_t .$$

where $\{M_t|t \in \mathcal{T}\}$ is an $\mathfrak{R}^N$-valued square-integrable martingale with respect to the natural filtration generated by $X$ under $\mathcal{P}$.

We now define the price dynamics of the risk-free bond and the investment fund. Firstly, we assume that the bond price process $\{B_t|t \in \mathcal{T}\}$ is governed by

$$dB_t = rB_t dt \quad t \in \mathcal{T}, \quad B_0 = 1 .$$

Here, the instantaneous market interest rate of the bond is assumed to be a positive constant, i.e., $r > 0$.

Let $W := \{W_t|t \in \mathcal{T}\}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$ with respect to its right-continuous, $\mathcal{P}$-completed, natural filtration. We suppose that $W$ and $X$ are stochastically independent under $\mathcal{P}$. The dynamics of the investment fund $S := \{S_t|t \in \mathcal{T}\}$ is governed by a Markovian regime-switching geometric Brownian motion:

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, \quad t \in \mathcal{T}, \quad S_0 = s .$$

The appreciation rate $\mu_t$ and the volatility $\sigma_t$ are modulated by the chain $X$ as follows:

$$\mu_t := \langle \mu, X_t \rangle, \quad \sigma_t := \langle \sigma, X_t \rangle, \quad t \in \mathcal{T} .$$

\(^1\)As explained in Boyle and Draviam (2007), if at time $t$, $X(t) = e_i$, then the state at $t + \Delta t$ will be $e_j$ with probability $a_{ij}\Delta t$, and the probability of remaining in the state is $1 - \sum_{j=0, j \neq i}^M a_{ij}\Delta t = 1 + a_{ii}\Delta t$.

\(^2\)This assumption ensures the regime switching P.D.E. can be obtained when pricing the dynamic fund protection.
Here $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^N$. Moreover, $\mu = (\mu_1, \mu_2, \ldots, \mu_N)' \in \mathbb{R}^N$ and $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)' \in \mathbb{R}^N$ for each $t \in \mathcal{T}$, where $\sigma_i > 0$ for all $i = 1, 2, \ldots, N$.

Let $\{Y_t | t \in \mathcal{T}\}$ be the logarithmic return process, where $Y_t := \ln(S_t/S_0)$ for each $t \in \mathcal{T}$. Then, the underlying fund dynamics can be written as:

$$S_t = S_0 e^{Y_t}, \quad t \in \mathcal{T},$$

where

$$Y_t = Y_0 + \int_0^t \left( \langle \mu, X_s \rangle - \frac{1}{2} \langle \sigma^2, X_s \rangle \right) ds + \int_0^t \langle \sigma, X_s \rangle dW_s, \quad t \in \mathcal{T}.$$  

Here, $\sigma^2 := (\sigma_1^2, \sigma_2^2, \ldots, \sigma_N^2)' \in \mathbb{R}^N$. For notational simplicity, write $\eta_i = \mu_i - \frac{1}{2} \sigma_i^2$ for $i = 1, 2, \ldots, N$ and $\eta = (\eta_1, \eta_2, \ldots, \eta_N)'$. Then, the dynamics of $Y$ can be rewritten as follows:

$$Y_t = Y_0 + \int_0^t \langle \eta, X_s \rangle ds + \int_0^t \langle \sigma, X_s \rangle dW_s, \quad t \in \mathcal{T}. \quad (6.2.1)$$

Let $\mathcal{F}^Y = \{\mathcal{F}^Y_t | t \in \mathcal{T}\}$ and $\mathcal{F}^S = \{\mathcal{F}^S_t | t \in \mathcal{T}\}$ be the natural filtrations generated by $\{Y_t | t \in \mathcal{T}\}$ and $\{S_t | t \in \mathcal{T}\}$, respectively. Also, we assume that the filtration given above are right continuous and complete. Since $\mathcal{F}^Y$ and $\mathcal{F}^S$ are equivalent, either of them could be used as the observed information structure. Here, we adopt $\mathcal{F}^Y$ as the observed information structure. We define the filtration $\mathcal{G} = \{\mathcal{G}_{t,s} | 0 \leq s \leq t \in \mathcal{T}\}$ by letting the double indexed $\sigma$-field $\mathcal{G}_{t,s} := \mathcal{F}_t^X \vee \mathcal{F}_s^Y$, for any $s, t \in \mathcal{T}$ with $s \leq t$, where $\mathcal{A} \vee \mathcal{B}$ is the minimal $\sigma$-algebra containing the $\sigma$-algebras $\mathcal{A}$ and $\mathcal{B}$. We write $\mathcal{G}_t = \mathcal{G}_{t,t}$, for all $t \in \mathcal{T}$, and $\mathcal{G} = \{\mathcal{G}_t | t \in \mathcal{T}\}$. Then $\mathcal{F}^Y$, $\mathcal{F}^X$ and $\mathcal{G}$ represent the flows of observable information, hidden information and full information, respectively.
6.3 The Esscher transform

Following Elliott et al. (2005), let $\theta_t$ be the regime switching Esscher parameter, which can be written as

$$\theta_t = \langle \theta, X_t \rangle,$$

where $\theta = (\theta_1, \theta_2, \ldots, \theta_N)'$. Write $(\theta \cdot Y)_t := \int_0^t \theta_s dY_s$, for each $t \in \mathcal{T}$. The regime switching Esscher transform $Q_\theta \sim \mathcal{P}$ on $\mathcal{G}_t$ with respect to a family of parameters $\{\theta_s|0 \leq s \leq t\}$ is given by

$$\frac{dQ_\theta}{d\mathcal{P}}|_{\mathcal{G}_t} = e^{(\theta \cdot Y)_t},$$

where $E[\cdot]$ is the expectation with respect to $\mathcal{P}$.

Since

$$E[e^{(\theta \cdot Y)_t}|\mathcal{F}_t^X] = \exp \left[ \int_0^t \left( \langle \mu, X_s \rangle - \frac{1}{2} \langle \sigma^2, X_s \rangle \right) \theta_s ds + \frac{1}{2} \int_0^t \langle \sigma^2, X_s \rangle \theta_s^2 ds \right],$$

the Radon-Nikodym derivative of the regime switching Esscher transform is given by

$$\frac{dQ_\theta}{d\mathcal{P}}|_{\mathcal{G}_t} = \exp \left\{ \int_0^t \langle \sigma, X_s \rangle \theta_s dW_s - \frac{1}{2} \int_0^t \langle \sigma^2, X_s \rangle \theta_s^2 ds \right\}.$$

Let $\{\theta_t^*|t \in \mathcal{T}\}$ be a family of risk-neutral regime switching Esscher parameters. Consider the following martingale condition:

$$S_0 = E_{Q_{\theta_t^*}} \left[ e^{-rt} S_t | \mathcal{F}_t^X \right], \quad t \in \mathcal{T}.$$

Using Bayes’ rule, the martingale condition can be rewritten as:

$$\int_0^t \left( \langle \mu, X_s \rangle - r + \theta_s^* \langle \sigma^2, X_s \rangle \right) ds = 0.$$

Elliott et al. (2005) shows that $\theta_t^*$ can be determined uniquely as:

$$\theta_t^* = \frac{r - \langle \mu, X_t \rangle}{\langle \sigma^2, X_t \rangle},$$

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Then, the Radon-Nikodym derivative can be written by:

\[
\frac{dQ_{\theta^*}}{dP} |_{\mathcal{G}_t} = \exp \left\{ \int_0^t \left( \frac{r - \langle \mu, X_s \rangle}{\langle \sigma, X_s \rangle} \right) dW_s - \frac{1}{2} \int_0^t \left( \frac{r - \langle \mu, X_s \rangle}{\langle \sigma, X_s \rangle} \right)^2 ds \right\} .
\]

Using Girsanov’s theorem, \( \tilde{W}_t = W_t + \int_0^t \left( \frac{r - \mu_s}{\sigma_s} \right) ds \) is a standard Brownian motion with respect to \( \mathcal{G} \) under \( Q_{\theta^*} \). Hence, under the probability measure \( Q_{\theta^*} \), the stock price dynamics can be written as

\[
dS_t = rS_t dt + \langle \sigma, X_t \rangle S_t d\tilde{W}_t .
\] (6.3.1)

### 6.4 Pricing the dynamic fund protection

During the whole investment period, investors who own the investment fund are guaranteed a predetermined protection level under the dynamic fund protection (DFP). Let \( K \) denote the constant protection level. As indicated by Imai and Boyle (2001) and Wong and Chan (2007), the amount received by the holder of the fund is then given by:

\[
S_T \max \left\{ 1, \max_{0 \leq \tau \leq T} \frac{K}{S_\tau} \right\} ,
\]

where \( S_t \) is the value of the fund without the protection. Hence, the terminal payoff for the DFP should be:

\[
\text{DFP}_T = S_T \max \left\{ 1, \max_{0 \leq \tau \leq T} \frac{K}{S_\tau} \right\} - S_T
\]

\[
= S_T \max \left\{ 0, \max_{0 \leq \tau \leq T} \frac{K}{S_\tau} - 1 \right\} .
\]

Write

\[
F_t = K / S_t \quad \text{and} \quad N_t = \max_{0 \leq \tau \leq t} F_\tau .
\]
Then

\[ \text{DFP}_T = S_T \max(N_T - 1, 0) . \]

As noted in Wong and Chan (2007), the payoff function of a fixed strike lookback call is:

\[ c_{\text{fix}}(T, F_T, N_T, 1) = \max(N_T - 1, 0) . \]

According to Wong and Chan (2007), if we view \( S \) as an exchange rate and hence \( F \) as an asset trading in the foreign currency world, this option can simply be valued as a fixed strike lookback call in “the foreign currency world” followed by multiplying the exchange rate \( S_t \). Note that the fixed strike price of the lookback call is one unity. Hence, a model-independent formula can be given:

\[ \text{DFP}_t = S_t \times c_{\text{fix}}(t, F_t, N_t) . \]

To value the DFP, we focus on the process of \( F \) in the foreign currency world. The dynamics of \( F_t \) under the risk-neutral probability \( Q_{\theta^*} \) is given by:

\[ dF_t = -(r - \sigma_t^2)F_t dt - \sigma_t F_t d\hat{W}_t . \] (6.4.1)

We now price the European-type fixed strike lookback call under a hidden Markov-modulated regime-switching model. Under the risk-neutral probability measure \( Q_{\theta^*} \), we define the price at time \( t \) with payoff \( V_T \) at maturity \( T \) as follows:

\[ V_t = \mathbb{E}_{Q_{\theta^*}} \left[ e^{-r(T-t)} V_T \mid \mathcal{G}_t \right] . \] (6.4.2)

Defining \( D_t := \max_{0 \leq u \leq t} S_u \), Shreve (2004) showed that the pair of processes \((S_t, D_t)\) has the Markov property, it is easy to show that processes \((F_t, N_t)\) also have the Markov property. There must exist \( v(t, x, y, z) \) such that \( V_t = v(t, X_t, F_t, N_t) \).
Theorem 6.4.1. Let \( v(t,x,y,z) \) denote the price of the fixed strike lookback option at time \( t \) under the assumptions that \( X_t = x, F_t = y, \) and \( N_t = z \). Note that \( \frac{\partial v}{\partial t}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z} \) and \( \frac{\partial^2 v}{\partial y^2} \) represent the first derivative of the function with respect to \( t, F_t \) and \( N_t \) and the second derivative with respect to \( F_t \) respectively. Then the price of the option is a solution of the following P.D.E:

\[
-rv(t,x,y,z) - \frac{\partial v}{\partial y}(t,x,y,z)(r - \sigma_t^2)y + \frac{\partial v}{\partial t}(t,x,y,z)
+ \frac{1}{2} \frac{\partial^2 v}{\partial y^2}(t,x,y,z)\sigma_t^2 y^2 + \langle v, Ax \rangle = 0
\]  

(6.4.3)
in the region of \( \{(t,x,y,z); 0 \leq t < T, 0 \leq y \leq z \} \). Here, \( v = (v_1, v_2, \ldots, v_N)' \).

Proof. The proof is standard. Applying the Itô’s differentiation rule to \( e^{-rt}v(t,X_t,F_t,N_t) \) gives:

\[
\begin{align*}
\frac{d}{dt} (e^{-rt}v(t,x,y,z)) &= e^{-rt} \left[ -rv(t,x,y,z)dt + \frac{\partial v}{\partial y}(t,x,y,z)dF_t + \frac{\partial v}{\partial t}(t,x,y,z)dt \\
&\quad + \frac{\partial v}{\partial z}(t,x,y,z)dN_t + \frac{1}{2} \frac{\partial^2 v}{\partial y^2}(t,x,y,z)d\langle F^c, F^c \rangle_t \\
&\quad + \langle v, Ax \rangle dt + \langle v, dM_t \rangle \right] \\
&= e^{-rt} \left[ -rv(t,x,y,z) - \frac{\partial v}{\partial y}(t,x,y,z)(r - \sigma_t^2)y + \frac{\partial v}{\partial t}(t,x,y,z) \\
&\quad + \frac{1}{2} \frac{\partial^2 v}{\partial y^2}(t,x,y,z)\sigma_t^2 y^2 + \langle v, Ax \rangle \right] dt \\
&\quad + e^{-rt} \frac{\partial v}{\partial z}(t,x,y,z) dN_t + e^{-rt}\langle v, dM_t \rangle \\
&\quad - e^{-rt} \sigma_t y \frac{\partial v}{\partial y}(t,x,y,z)d\bar{W}_t.
\end{align*}
\]

To ensure that the discounted value process is a martingale, the drift term should be zero, i.e., the coefficients of the \( dt \) term must be zero. This gives us the martingale
condition. The system of partial differential equations needs to be solved backwardly subject to particular terminal conditions and the boundary conditions.

It is difficult to determine the boundary conditions for the fixed strike lookback call. Here, similar in Wong and Chan (2007), we first focus on the lookback options with linear homogeneous payoffs, i.e., the floating strike lookback options. They stated all existing lookback options could be valued through the pricing formulas of floating strike lookback options. Besides, Wong and Kwok (2003) showed put-call parity relations between a fixed strike lookback call and a float strike lookback put option. Then we could derive the price of fixed lookback call options.

The corresponding terminal payoff of the floating strike lookback put is given by

\[ v(T, x, y, z) = z - y \]

Following Wilmott et al. (1997), Boyle and Draviam (2007), the boundary conditions are

\[ v(t, x, 0, z) = ze^{-r(T-t)} \]

and

\[ \frac{\partial v}{\partial z}(t, x, y = z) = 0 \]

6.5 Estimation of HMM parameters

Since the hidden state of the economy completely determines the unknown parameters, the state of the hidden Markov model \( X_t \) plays a vital role in our model. Note that there are two sources of uncertainty, in our model, leading to an incomplete market. For option valuation we discussed earlier, we consider a risk-neutral probability \( Q_\theta \) selected
by the Esscher transform. Siu (2011) showed that an optimal equivalent martingale measure selected by the minimal relative entropy approach does not price the regime-switching risk. Then, the probability law of the chain $X$ remains the same after the measure change. It is not unreasonable to consider the hidden states of the Markov chain under the real-world probability $\mathcal{P}$ instead of the risk-neutral probability. In practice, the return process is observed under $\mathcal{P}$ instead of under $Q$. Consequently, it is reasonable to discuss the estimation of the Markov chain parameters under $\mathcal{P}$ rather than under $Q$. Note that real-data are collected in discrete-time. Consequently, to estimate the parameters of the hidden Markov chain, algorithms based on discretization of continuous-time filters and smoothers are needed. Here, we adopt the well-known the Baum-Welch algorithm, based on the discretization of continuous-time smoothers\footnote{As discussed in earlier works, compared to filtered estimates, the smoothed estimates can incorporate the extra information obtained from the observations between time $t$ and $T$. Furthermore, to apply the robust discretization of continuous-time filters, the dynamics of the observation process should not include stochastic integrals. This is also the reason why Elliott et al. (2003) made approximations to the observation process. For details, interested readers can refer to Elliott et al. (1994), Elliott et al. (2003), James et al. (1996), Malcolm and Elliott(2010).}.

### 6.5.1 Estimating states of the hidden Markov model

The smoother-based estimators facilitate the application of the EM algorithm, which can be regarded as an extension of the discrete-time Baum-Welch algorithm. However, it is worth mentioning that the Baum-Welch algorithm only finds a local maximum, rather than a global maximum, in the parameter space. Here, we will give a brief introduction to the Baum-Welch algorithm. Let $P = [p_{ij}]_{i,j=1,2,...,N}$ be the transition probability matrix, where $p_{ij} = P(X_t = e_j | X_{t-1} = e_i)$ and $\pi_i = P(X_1 = e_i)$ represent
the initial state distribution. In addition, the corresponding observation sequence is assumed to be \( Y_1 = y_1, \ldots, Y_T = y_T \). Let \( c_j(y_t) = P(Y_t = y_t | X_t = e_j) \) be the probability of a particular observation at time \( t \) when the Markov chain is in the state \( e_j \) and \( C(\cdot) = (c_1(\cdot), c_2(\cdot), \ldots, c_N(\cdot)) \). Specially, we partition the observation sequence into \( L \) groups, i.e., \( O = \{o_1, \ldots, o_L\} \). As assumed in Bilmes (1998), we write the complete set of HMM parameters as \( \psi = (P, C, \pi) \) \(^4\).

**Step 1:** Firstly, we give a brief introduction to the forward and backward procedures given in Bilmes (1998). Define the following forward variable \( \alpha_i(t) \) and backward variable \( \beta_i(t) \)

\[
\alpha_i(t) = P(Y_1 = y_1, \ldots, Y_t = y_t, X_t = e_i | \psi),
\beta_i(t) = P(Y_{t+1} = y_{t+1}, \ldots, Y_T = y_T | X_t = e_i, \psi).
\]

Here, \( \alpha_i(t) \) is the probability of observing \( (y_1, \ldots, y_t) \) and the state of the Markov chain is \( e_i \) at time \( t \). \( \beta_i(t) \) is the probability of observing \( (y_{t+1}, \ldots, y_T) \) when the state of the Markov chain is \( e_i \) at time \( t \). Then, as in Bilmes (1998), an efficient induction of the pair \( \alpha_i(t) \) and \( \beta_i(t) \) is as follows:

\[
\alpha_i(1) = \pi_i c_i(y_1),
\alpha_j(t + 1) = \left[ \sum_{i=1}^{N} \alpha_i(t) p_{ij} \right] c_j(y_{t+1}),
\]

\(^4\)Let \( \Psi \) be an open subset of \( \mathbb{R}^n \). Suppose that to each \( \psi \in \Psi \), we have a smooth assignment \( \psi \rightarrow (P(\psi), B(\psi), \pi(\psi)) \). Under these assumptions, for each fixed \( y_1, y_2, \ldots, y_T \), \( P_{y_1,y_2,\ldots,y_T}(\psi) = P_{y_1,y_2,\ldots,y_T}(P(\psi), B(\psi), \pi(\psi)) \) is a smooth function of \( \psi \). Given a fixed \( Z \)-sample \( y = y_1, \ldots, y_T \), a parameter value \( \psi^* \) which maximizes the likelihood \( P_y(\psi) = P_{y_1,\ldots,y_T}(\psi) \) can be found. To maximize the likelihood function, Baum et al. (1970) defined a continuous transformation \( T \) mapping \( \Psi \) into itself with the property that \( P_{y_1,y_2,\ldots,y_T}(T(\psi)) > P_{y_1,y_2,\ldots,y_T}(\psi) \) unless \( \psi \) is a critical point of \( P_{y_1,y_2,\ldots,y_T}(\psi) \).
\[ P(Y|\psi) = \sum_{i=1}^{N} \alpha_i(T), \]

and

\[ \beta_i(T) = 1, \]

\[ \beta_i(t) = \sum_{j=1}^{N} p_{ij} c_j(y_{t+1}) \beta_j(t+1), \]

\[ P(Y|\psi) = \sum_{i=1}^{N} \beta_i(1) \pi_i c_i(y_1). \]

**Step 2:** Then, another two random variables are defined. The first is the probability of the chain being in state \( e_i \) at time \( t \) given the observation sequence \( Y \).

\[ \gamma_i(t) = P(X_t = e_i|Y, \psi) = \frac{P(Y, X_t = e_i|\psi)}{P(Y|\psi)} = \frac{P(Y, X_t = e_i|\psi)}{\sum_{j=1}^{N} P(Y, X_t = e_j|\psi)}. \]

Note that

\[ \alpha_i(t) \beta_i(t) = P(y_1, \ldots, y_t, X_t = e_i|\psi) P(y_{t+1}, \ldots, y_T|X_t = e_i, \psi) = P(Y, X_t = e_i|\psi), \]

so \( \gamma_i(t) \) can be rewritten as

\[ \gamma_i(t) = \frac{\alpha_i(t) \beta_i(t)}{\sum_{j=1}^{N} \alpha_j(t) \beta_j(t)}. \]

The second one is the probability of the chain being in state \( e_i \) at time \( t \) while being in state \( e_j \) at time \( t + 1 \).

\[ \xi_{ij}(t) = P(X_t = e_i, X_{t+1} = e_j|Y, \psi) \]

\[ = \frac{P(X_t = e_i, X_{t+1} = e_j, Y|\psi)}{P(Y|\psi)} \]

\[ = \frac{\alpha_i(t) p_{ij} c_j(y_{t+1}) \beta_j(t+1)}{\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i(t) p_{ij} c_j(y_{t+1}) \beta_j(t+1)}. \]

It is worth noting that the sum of the above two variables have practical interpretations.
(a) $\sum_{t=1}^{T} \gamma_i(t)$: the expected number of times in state $e_i$,

(b) $\sum_{t=1}^{T-1} \xi_{ij}(t)$: the expected number of transitions from state $e_i$ to state $e_j$ for $Y$.

**Step 3:** Then, the informal version of the Baum-Welch algorithm \(^5\) is derived.

(a) $\bar{\pi}_i = \gamma_i(1)$: the expected relative frequency spent in state $e_i$ at time 1.

(b) $\bar{a}_{ij} = \frac{\sum_{t=1}^{T-1} \xi_{ij}(t)}{\sum_{t=1}^{T} \gamma_i(t)}$: the expected number of transitions from state $e_i$ to state $e_j$ relative to the expected total number of transitions away from state $e_i$.

(c) $\bar{b}_i(k) = \frac{\sum_{t=1}^{T} 1_{y_t = o_k} \gamma_i(t)}{\sum_{t=1}^{T} \gamma_i(t)}$: the expected number of times the output observations have been equal to $o_k$ while in state $e_i$ relative to the expected total number of times in state $e_i$.

**Step 4:** Here, we choose the Viterbi algorithm to determine the most likely path, i.e., the path $(q_1, \ldots, q_T)$ which maximizes the likelihood $p(q_1, q_2, \ldots, q_T | Y, \psi)$.

Let $\delta_i(t)$ be the path which is in state $e_i$ at time $t$ with the highest probability, i.e.,

$$\delta_i(t) = \max_{q_1, q_2, \ldots, q_{t-1}} P(q_1, q_2, \ldots, q_t = e_i, y_1, y_2, \ldots, y_t | \psi),$$

then

$$\delta_j(t + 1) = \max_i [\delta_i(t)p_{ij}] c_j(y_{t+1}).$$

The recursive form of the variable is

$$\delta_i(1) = \pi_i c_i(y_1),$$

\(^5\) A more general case is that the underlying state sequence is assumed to be hidden or unobserved. In this case, the $Q$ function is introduced. Details can refer to Baum et al. (1970).
\[ \delta_j(t) = \max_{1 \leq i \leq N} \delta_i(t - 1) p_{ij} c_j(y_t), \]

and the most probable path satisfies the following conditions

\[ q_t = e_{i^*}, \quad t = T, T - 1, T - 2, \ldots, 1, \]

where

\[ i^* = \begin{cases} 
\arg \max_{1 \leq i \leq N} \delta_i(T) & t = T, \\
\arg \max_{1 \leq i \leq N} \delta_i(t) P(X_{t+1} = q_{t+1} | X_t = e_i), & t = T - 1, T - 2, \ldots, 1. 
\end{cases} \]

After we find the most-probable path through the HMM states using the Viterbi approach, we can estimate the parameters of the dynamics of the investment fund.

### 6.5.2 Maximum likelihood estimations of parameters

When we obtain the most-probable path, we further use the maximum likelihood estimation approach to estimate the appreciation rate and the volatility in the dynamics of asset price.

It is easy to see that the probability density function of the observation process \( Y \) is given by

\[
p(Y_t = y_t | \eta_1, \eta_2, \ldots, \eta_N, \sigma_1, \sigma_2, \ldots, \sigma_N) = \sum_{j=1}^{N} \frac{1}{\sqrt{2\pi\sigma_j^2}} \exp \left( -\frac{(y_t - \eta_j)^2}{2\sigma_j^2} \right) \mathbb{1}_{\{q_t = e_j\}}.
\]

Then the likelihood function is

\[
\mathcal{L}(y_1, \ldots, y_T; \eta_1, \eta_2, \ldots, \eta_N, \sigma_1, \ldots, \sigma_N) = \prod_{t=1}^{T} \sum_{j=1}^{N} \frac{1}{\sqrt{2\pi\sigma_j^2}} \exp \left( -\frac{(y_t - \eta_j)^2}{2\sigma_j^2} \right) \mathbb{1}_{\{q_t = e_j\}}.
\]
Let the derivatives of the logarithm likelihood function with respect to $\eta_j$ and $\sigma_j$ ($j = 1, \ldots, N$) be zero. The maximum likelihood estimations of the parameters can be derived.

\[
\hat{\eta}_j = \frac{\sum_{t=1}^{T} y_t \mathbbm{1}_{\{q_t = e_j\}}}{\sum_{t=1}^{T} \mathbbm{1}_{\{q_t = e_j\}}} ,
\]

\[
\hat{\sigma}_j^2 = \frac{\sum_{t=1}^{T} (y_t - \hat{\eta}_j)^2 \mathbbm{1}_{\{q_t = e_j\}}}{\sum_{t=1}^{T} \mathbbm{1}_{\{q_t = e_j\}}} \quad \text{for} \quad j = 1, 2, \ldots, N .
\]

**Remark 6.5.1.** One may argue that the smoother-based estimation equations are recursive and can be implemented by the standard method of discretization. However, when the states of the hidden Markov chain and the observation sequences are discrete, forward-backward estimates are comparatively easier to be calculated under the Baum-Welch method. Furthermore, the EM algorithm concerning the smoother-based estimation requires initial estimates of both the transition matrix of the hidden Markov chain and the volatilities of the observation process. It is reasonable that the convergence of the algorithm depends on the selection of the initial estimates. Consequently, we determine the transition matrix and the states of the hidden Markov chain first. Then, we can estimate the appreciation rate and the volatility of the investment fund via the maximum likelihood estimation.

### 6.6 Numerical examples

When we obtain the estimates of the filtered hidden Markov chain, we can replace the corresponding hidden quantities by their filtered estimates. Then, the P.D.E equations
(6.4.2) are given as follows:

\[-rv(t, x, y, z) - \frac{\partial v}{\partial y}(t, x, y, z)(r - \hat{\sigma}^2) + \frac{\partial v}{\partial t}(t, x, y, z) + \frac{1}{2} \frac{\partial^2 v}{\partial y^2}(t, x, y, z)\hat{\sigma}^2 y^2 + \left\langle v, \hat{A}x \right\rangle = 0.

Here, we adopt similar dimension reduction technique as Wilmott et al. (1997) and Boyle and Draviam (2007). Let \( u = \frac{y}{z}, G_i(t, u) = \frac{1}{z} v(t, x = e_i, y, z). \) Then the P.D.E becomes

\[-\frac{\partial G_i}{\partial u}(t, u)(r - \hat{\sigma}_i^2)u + \frac{1}{2} \frac{\partial^2 G_i}{\partial u^2}(t, u)\hat{\sigma}_i^2 u^2 + \frac{\partial G_i}{\partial t}(t, u) + \left\langle G_i, \hat{A}e_i \right\rangle - rG_i(t, u) = 0.

The corresponding terminal conditions become the following form:

\[ G_i(t = T, u) = \max(1 - u, 0). \]

The boundary conditions are

\[ G_i(t, u = 0) = e^{-r(T-t)}, \]

and when the maximum is measured continuously, the boundary condition at \( F = N \) becomes a boundary condition at \( u = 1: \)

\[ \frac{\partial G_i}{\partial u} \bigg|_{u=1} = G_i(t, u = 1). \]

It is easy to solve the above P.D.E and we could obtain the prices of the floating lookback put options. According to the put-call parity in Wong and Kwok (2003),

\[ c_{fix}(t, q, F, N_i, ; 1) = p_{fl}(t, q, F, \max(N_i, 1)) + F - e^{-r(T-t)}, \]

we can derive the prices of the fixed strike lookback call options and the dynamic fund protection contract.
Here, we shall give two numerical examples to illustrate the valuation of dynamic fund protection when the dynamics of the investment fund follow a Geometric Brownian motion with parameters being modulated by a hidden Markov chain. We assume that the Markov chain has two states. The real data examples we present here use data sets of daily closing prices of IBM (International Business Machines Corporation) and Apple Inc., from May 2008 to April 2010, retrieved from Yahoo Finance. There are 484 observations in each data set. During the period, the economy was influenced by the financial crisis. Consequently, it is of practical value to investigate the valuation problem when the economy has substantial changes. On the other hand, the stock prices of financial industries have not been selected since the whole industry was affected significantly by the crisis. As mentioned earlier, the Baum-Welch algorithm is adopted. We derived the estimates of relevant parameters.

\[
\hat{P}_{IBM} = \begin{pmatrix} 0.7794 & 0.2206 \\ 0.1829 & 0.8171 \end{pmatrix}, \quad \hat{P}_{Apple} = \begin{pmatrix} 0.7982 & 0.2018 \\ 0.1982 & 0.8018 \end{pmatrix}.
\]

Here \( \hat{P}_{IBM} \) and \( \hat{P}_{Apple} \) represent the transition probability matrix of the IBM example and the Apple Inc. example respectively. From the given data, we can easily calculate the following estimates:

\[
\hat{\eta}_{IBM} = \begin{pmatrix} -0.0678 \\ 0.2148 \end{pmatrix}, \quad \hat{\eta}_{Apple} = \begin{pmatrix} 0.0828 \\ 0.6253 \end{pmatrix},
\]

\[
\hat{\sigma}^2_{IBM} = \begin{pmatrix} 0.0101 \\ 0.0013 \end{pmatrix}, \quad \hat{\sigma}^2_{Apple} = \begin{pmatrix} 0.0341 \\ 0.0100 \end{pmatrix}.
\]

Clearly, in the two examples, the estimators of the volatilities \( \sigma_1 > \sigma_2 \), indicate that State 1 and State 2 represent a “Bad” economy and a “Good” one, respectively. The estimation results are consistent with those in Liew and Siu (2010). The differences
between the estimated volatilities in the two states may be attributed to structural changes in data due to the financial crisis and the rebounds after the crisis. Note that the estimated drift parameter for a “Bad” economy in a regime-switching model is not uncommon, which is similar with the numerical example in Liew and Siu (2010).

After we obtain the estimates, we can solve the valuation problem using the partial differential equation approach. The values for parameters, used in the discretization of the partial differential equations, are assumed to be:

\[ \Delta t = \frac{1}{250} \text{(year)} \]

Also, we assume the risk-free interest rate is:

\[ r = 0.05 \text{(1/year)} \]

Using finite difference methods, we calculate the values of the DFP by varying the guarantee level \( K \) and the time-to-maturity \( T \). Here, we assume the value of \( K \) is less than the value of the naked fund on May 1st, 2008. It is not unreasonable to make this assumption. The organizations issuing the DFP will never design such investment plans, if so, they will upgrade the fund immediately after the DFP issued. Table 6.6.1 and Table 6.6.2 present numerical results for the price of a dynamic fund protection.

From Table 6.6.1 and Table 6.6.2, we note that the prices of DFP decrease when the maturity time is longer in the IBM example, while the opposite trend presents in the Apple Inc. example. However, as calculated earlier, the other parameters are similar in the two examples. This is also another reason why we chose the stock prices of the two companies. Note that in the Black-Scholes formula, the Greek “Theta” is defined to measure the change in the option price when the time to maturity decreases. Our results verify the result that “Theta” can be either positive or negative.

In addition, we simulate the price of the DFP under the Black-Scholes model. Goldman et al. (1979) derived the pricing formula for a floating strike lookback put
Table 6.6.1: DFP prices for IBM versus different K and T

<table>
<thead>
<tr>
<th>Protection level ($)</th>
<th>T=2 State 1</th>
<th>State 2</th>
<th>T=3 State 1</th>
<th>State 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>95</td>
<td>27.4092</td>
<td>7.1824</td>
<td>22.0819</td>
<td>4.2292</td>
</tr>
<tr>
<td>100</td>
<td>33.6141</td>
<td>11.0479</td>
<td>27.7742</td>
<td>8.0946</td>
</tr>
<tr>
<td>105</td>
<td>39.8190</td>
<td>14.9134</td>
<td>33.4664</td>
<td>11.9601</td>
</tr>
<tr>
<td>110</td>
<td>46.0239</td>
<td>18.7789</td>
<td>39.1586</td>
<td>15.8256</td>
</tr>
<tr>
<td>115</td>
<td>52.2288</td>
<td>22.7863</td>
<td>44.8509</td>
<td>19.7899</td>
</tr>
<tr>
<td>120</td>
<td>58.4337</td>
<td>27.7111</td>
<td>50.5431</td>
<td>24.3925</td>
</tr>
</tbody>
</table>

Table 6.6.2: DFP prices for Apple Inc. versus different K and T

<table>
<thead>
<tr>
<th>Protection level ($)</th>
<th>T=2 State 1</th>
<th>State 2</th>
<th>T=3 State 1</th>
<th>State 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>160</td>
<td>103.7537</td>
<td>22.5602</td>
<td>107.7473</td>
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</tr>
<tr>
<td>170</td>
<td>115.8936</td>
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<tr>
<td>180</td>
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<td>36.6907</td>
<td>131.9745</td>
<td>40.8538</td>
</tr>
<tr>
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<td>140.1732</td>
<td>43.7559</td>
<td>144.0882</td>
<td>47.9052</td>
</tr>
<tr>
<td>200</td>
<td>152.3131</td>
<td>50.8212</td>
<td>156.2018</td>
<td>54.9566</td>
</tr>
<tr>
<td>210</td>
<td>164.4529</td>
<td>57.8864</td>
<td>168.3154</td>
<td>62.0079</td>
</tr>
</tbody>
</table>
Figure 6.6.1: DFP prices corresponding to different protection level when $T = 2$ and $T = 3$

option. Then we calculated the prices of the corresponding DFPs. Here, we only give the comparison results in the IBM example. Similar results can be simulated in the Apple Inc. example. In Fig. 6.6.1, two solid lines indicate values arising from our approach while two dashed lines correspond to the BS model. And the lines with * represent the prices of DFP are derived when the economy state is ”Good”. Besides, from the numerical results, the effect of different economy states cannot be ignored in the valuation process of DFP.

To highlight the effect of regime-switching parameters in the DFP valuation, we construct Table 6.6.3 that compares the BS price and our price numerically. Here, we simply take the prices when $T = 2$ as an example. As seen from Table 6.6.3, when volatility changes, the prices of DFP change significantly. It means that the price of a DFP will significantly change if the state of the economy changes. Thus, it is not hard to imagine the dilemma faced by insurance companies. If an investment plan is sold at a lower price, insurance companies may face large losses when the economy is
Table 6.6.3: DFP prices for IBM under different models with $T=2$

<table>
<thead>
<tr>
<th>Protection level ($)</th>
<th>BS model State 1</th>
<th>BS model State 2</th>
<th>Our model State 1</th>
<th>Our model State 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>95</td>
<td>24.3810</td>
<td>6.0101</td>
<td>27.4092</td>
<td>7.1824</td>
</tr>
<tr>
<td>100</td>
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<td>10.8379</td>
<td>33.6141</td>
<td>11.0479</td>
</tr>
<tr>
<td>105</td>
<td>38.7840</td>
<td>15.8445</td>
<td>39.8190</td>
<td>14.9134</td>
</tr>
<tr>
<td>110</td>
<td>46.1506</td>
<td>20.8948</td>
<td>46.0239</td>
<td>18.7789</td>
</tr>
<tr>
<td>115</td>
<td>53.6273</td>
<td>25.9613</td>
<td>52.2288</td>
<td>22.7863</td>
</tr>
<tr>
<td>120</td>
<td>61.2141</td>
<td>31.0896</td>
<td>58.4337</td>
<td>27.7111</td>
</tr>
</tbody>
</table>

“bad”. Consequently, insurance companies would like to charge a higher price to hedge the volatility risk. However, on the other hand, the DFP with a higher price would be less attractive to an investor. To some extent, this explains that the effect of hidden Markov model cannot be ignored in the valuation process.

To make a comparison between the estimates of the DFP price $\hat{DFP}$ and the Black-Scholes price $DFP_{bs}$, we plot the relative difference between the Black-Scholes price and our estimation. Fig. 6.6.2 is the plot of relative difference $\frac{DFP_{bs} - \hat{DFP}}{DFP_{bs}}$ versus $K$, for several fixed values for $T$. In Fig. 6.6.2, the two solid lines represent the relative differences when the maturity time is $T = 2$ and the two dash lines depict the relative differences when $T = 2$. The lines with * illustrate that the economic state is “Good”. From the figure, it is clear that the differences between the estimated dynamic fund protection values and the BS ones are comparatively small, say between 0 and 0.5% most of the time. Moreover, the larger the protection level, the smaller the relative differences are. In summary, the estimates $\hat{DFP}$ provide a reasonably good
Figure 6.6.2: Relative differences $\frac{DFP_{bs} - \tilde{DFP}}{DFP_{bs}}$ versus $K$

approximation to the BS dynamic fund protection estimates.

6.7 Conclusions

This paper considered the valuation of the dynamic fund protection (DFP). We assumed that the investment fund depends upon a continuous-time, finite-state, hidden Markov chain. Firstly, Esscher transform was applied to select a pricing kernel. Then, we adopted the partial differential equation approach to price the DFP. To estimate the states and parameters of the hidden Markov model, Baum-Welch algorithm, based on the discretization of continuous-time smoothers, was applied. After we obtained the most-likely path of the hidden Markov model by the Viterbi algorithm, we derived the maximum likelihood estimation of the model parameters. During the numerical examples, we calculated the prices of the DFP for two companies, IBM and Apple Inc.
Chapter 7

Conclusions

In this thesis, we have investigated the valuation of several financial products, including European-style options, foreign equity options, power options, equity-linked products, dynamic fund protection plans, under different regime-switching models. Regime-switching diffusion models and jump-diffusion models are mainly considered in this thesis. We assume the Markov chain is observable in Chapter 2-Chapter 5, while the Markov chain is unobservable in Chapter 6. In Chapter 6, we adopted a three-stage estimation approach to estimate the parameters of the unobservable Markov chain and the model dynamics.

However, there exist many possible extensions of this thesis. In this thesis, we mainly focus on the valuation of financial derivatives and insurance products with embedded option features under regime-switching models. It is of great practical value to consider the hedging of these products under regime-switching models. Static hedging and dynamic hedging are two main types of hedging. We could investigate the static hedging and the dynamic hedging of standard options, exotic options and insurance products with embedded option features and provide comparisons of these
hedging strategies. Other possible directions include indifference pricing of financial derivatives under regime-switching models, valuation of financial derivatives with the minimal martingale entropy measure, relationships between the equivalent martingale measures selected by the approaches mentioned in Chapter 1, valuation of interest rate derivatives via different approaches under regime-switching models, etc.
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