Relation between classical communication capacity and entanglement capability for two-qubit unitary operations

Dominic W. Berry and Barry C. Sanders

Department of Physics and Centre for Advanced Computing—Algorithms and Cryptography, Macquarie University, Sydney, New South Wales 2109, Australia

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Two-qubit operations may be characterized by their capacities for communication, both with and without free entanglement, and their capacity for creating entanglement. We establish a set of inequalities that give an ordering to the capacities of two-qubit unitary operations. Specifically, we show that the capacities for entanglement creation and bidirectional communication without entanglement assistance are at least as great as half the bidirectional communication capacity with entanglement assistance. In addition, we show that the bidirectional communication that can be performed using an ensemble may be increased via a two-qubit unitary operation by twice the operation’s capacity for entanglement.

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I. INTRODUCTION

Quantum information processing relies on applications of single-qubit unitary transformations and nonlocal two-qubit unitary transformations, where the term “nonlocal” refers to joint operations on separate Hilbert spaces. These nonlocal operations are powerful tools in quantum information theory and can be characterized by the entanglement they can generate between the two Hilbert spaces as well as by the communication capacity they can deliver. These two seemingly disparate measures of gate strengths are useful for quite different applications of two-qubit nonlocal gates. However, we establish that these two capacities are closely related.

Entanglement capability has been investigated in detail [1–9]. The entanglement increase for finite unitaries has been analyzed for initially unentangled states [5] and investigated by numerical methods for initially entangled states [6]. An important feature of the entanglement capability is that it quantifies the “strength” of the gate, which is particularly useful in considering the resource requirements for simulation of specific gate operations or Hamiltonian dynamics by other gates [10].

The communication capacity of nonlocal unitary transformations is as important. The communication capacity is relevant to performing quantum gates remotely using entanglement and classical communication [1,11–14]. An early example of the remarkable relation between classical communication and entanglement was seen in superdense coding [15].

It is known that there is a relationship between entanglement capability and communication capacity for some simple gates. For example, the CNOT gate can create one ebit [the entanglement inherent in an ideal Einstein-Podolsky-Rosen (EPR), or Bell, pair] of entanglement between two initially unentangled systems; the CNOT gate can thus be used to generate a Bell pair. In addition, the CNOT gate can be used to communicate one bit of information in each direction simultaneously [12,13]. The SWAP gate may create two ebits of entanglement or perform two bits of communication in one or both directions simultaneously [12,13,15]. In both cases the communication is achieved while consuming entanglement.

The relationship between communication capacity and entanglement capability has been addressed by Bennett, Harrow, Leung, and Smolin (BHLS) [16]. They define asymptotic entanglement capabilities and communication capacities for unitary operations and derive many relations for these quantities.

Our goal here is to establish rigorous relations between entanglement capability and communication capacity for two-qubit unitary operations. By investigating the entanglement generated by a superposition of classical messages, we recently showed that the entanglement capability of two-qubit unitary operations is at least as great as half the capacity for error-free entanglement-assisted communication [17]. Here we provide an extended version of those results, with rigorous derivations for nonzero error rate.

In Sec. II we review the definitions of Ref. [16], with minor modifications, and introduce the notation used throughout the paper. We show that the communication rates may be achieved with an upper bound on the dimension of the ancilla in Sec. III. We then apply this result to prove an inequality between the entanglement-assisted communication capacity and the capacities for creating and destroying entanglement (Sec. IV A).

The results presented up to Sec. IV A may be applied to all bipartite unitary operations. We then restrict to two-qubit unitaries, and in Sects. IV B to VII establish a sequence of inequalities between the various capacities for two-qubit unitaries. We discuss a possible method of proving equalities in Sec. VII, and conclude in Sec. VIII.

II. CAPACITY DEFINITIONS

The asymptotic definitions of capacities for entanglement creation and communication were introduced by BHLS. In this section we summarize the definitions of BHLS, and make some minor improvements. We consider two parties, Alice and Bob, with the combined Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. We call a unitary operator $U$ acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ local if it is of the form $V_A \otimes V_B$, where $V_A$ and $V_B$ act on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Operators that are not of this form we will term nonlocal. (This is not meant to imply physical separation between the physical systems represented by $\mathcal{H}_A$ and $\mathcal{H}_B$.)
In general, the Hilbert spaces $H_A$ and $H_B$ will be of the form

$$H_A = H_{A_{\text{anc}}} \otimes H_{A_1}, \quad H_B = H_{B_{\text{anc}}} \otimes H_{B_1},$$

such that $U$ acts only upon $H_{A_{\text{anc}}} \otimes H_{B_{\text{anc}}}$. The Hilbert spaces $H_{A_{\text{anc}}}$ and $H_{B_{\text{anc}}}$ will be called the ancillas.

The maximum increase in entanglement that may be produced by a single implementation of the unitary transformation $U$ is denoted as $E_U$. That is, we define

$$E_U = \sup_{\Psi} \frac{1}{H_A \otimes H_B} \left[ E(U|\Psi) - E(\Psi) \right].$$

The quantity $E(\cdots)$ is the entropy of entanglement $E(\{|\Psi\rangle\}) = S(\text{Tr}_A(|\Psi\rangle\langle\Psi|))$, where $S(\rho) = -\text{Tr}(\rho \log \rho)$. Throughout we employ logarithms to base 2, so the entanglement capability is maximized for pure states and is arbitrary large, but finite, dimension. The entanglement capability is quantified by the pair of numbers $(E_U, R_U)$.

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BHLS show that the asymptotic entanglement capability per operation in the limit of a large number of operations is equal to $E_U$. Similarly, $E_U$ may be interpreted as the asymptotic entanglement capability of $U$ [16].

To quantify the communication capacity of the unitary operation $U$, we adopt the definitions of asymptotic capacities introduced by BHLS. The bidirectional communication scheme is quantified by the pair of numbers $(R_-, R_-)$. The pair $(R_-, R_-)$ is said to be achievable if, for any $\epsilon > 0$, there exists $t$ such that it is possible to communicate $tR_--\epsilon$ bits from Alice to Bob and $tR_-\epsilon$ bits from Bob to Alice with fidelity $1 - \epsilon$ via $t$ applications of $U$ interspersed with local unitary operations.

To explain this communication scheme in more detail, Alice and Bob share the Hilbert space $H_{A_1} \otimes H_{B_1}$, where $H_{A_1} = H_{A_{\text{anc}}} \otimes H_{A_1}$ and $H_{B_1} = H_{B_{\text{anc}}} \otimes H_{B_1}$. This division of the Hilbert space should not be confused with that in Eq. (1); here we do not specify the component of the Hilbert space upon which $U$ acts. Subsystems $A_1$ and $B_1$ are both of dimension $2^{n_a}$. Subsystem $A_1$ is initially in one of $2^n$ orthogonal states corresponding to Alice’s $n_a$-bit message $x$, and $B_1$ is initially in a state corresponding to Bob’s $n_b$-bit message $y$. Subsystems $A_2$ and $B_2$ are the ancillas for Alice and Bob, respectively, and are initially in a shared state $|\psi\rangle_{A_2B_2}$. The complete initial state may be expressed as $|x\rangle_{A_1}|y\rangle_{B_1}|\psi\rangle_{A_2B_2}$.

The communication scheme consists of $t$ applications of $U$, interspersed with the intermediate local unitary operations $V_A^{(k)} \otimes V_B^{(k)}$. The final state is therefore

$$|\eta_{xy}\rangle_{AB} = (V_A^{(t)} \otimes V_B^{(t)}) U(V_A^{(t-1)} \otimes V_B^{(t-1)}) U \cdots U(V_A^{(0)} \otimes V_B^{(0)}) |x\rangle_{A_1} |y\rangle_{B_1} |\psi\rangle_{A_2B_2}.$$  

For perfect communication Bob’s message $y$ is transferred to $A_1$ and Alice’s message $x$ is transferred to $B_1$. For communication with fidelity at least $1 - \epsilon$, we have

$$F(|y\rangle_{A_1} |x\rangle_{B_1}, \text{Tr}_{A_2B_2} \langle \eta_{xy}\rangle_{AB} |\eta_{xy}\rangle_{AB}) \geq 1 - \epsilon,$$

where $F(|\psi\rangle, \rho) = (|\psi\rangle \langle \psi| \rho)$. This condition means that the classical error rate is no greater than $\epsilon$.

As in Ref. [16], we say that the pair $(R_-, R_-)$ is achievable if, for all $\epsilon > 0$, there exists a $t$ such that it is possible to perform communication of fidelity at least $1 - \epsilon$ in this way with $n_a \geq tR_-$ and $n_b \geq tR_-$. For communication that is not assisted by entanglement, we take the initial ancilla state $|\psi\rangle_{A_2B_2}$ to be an unentangled state of arbitrary finite dimension. For entanglement-assisted communication, we allow an ancilla state that is a tensor product of an unentangled state and a maximally entangled state, both of which may be of arbitrary finite dimension.

For the entanglement-assisted case, it is necessary to allow an unentangled component to the ancilla because the communication scheme is composed entirely of unitary operations. Some of the processes that communication schemes use require an unentangled subsystem, and it is not possible to obtain an unentangled subsystem using local unitary operations.

For example, consider the measurement of some state $|\psi\rangle$ via a $d$-element positive operator-valued measure $\{P_i\}$. In order to treat the measurement as a unitary process, we add a register of dimension $d$, which is initialized in an unentangled state $|n\rangle$. The measurement may then be represented by a unitary operation $U_{\text{meas}}$ such that $U_{\text{meas}} |n\rangle |\psi\rangle = \sum_i |n+\rangle P_i |\psi\rangle$, where $\oplus$ is addition modulo $d$. The register stores the result of the measurement, which is only possible because it is initially unentangled.

Thus it is necessary to have an unentangled subsystem in order to perform the unitary equivalent of local measurements. In particular, because the communication scheme considered by Ref. [16] uses local measurements, it requires an unentangled resource. It is therefore necessary to allow an unentangled component to the ancilla for our definition to be consistent with Ref. [16].

We denote the set of achievable rate pairs without entanglement assistance by $S_U$, and the set of achievable rate pairs with entanglement assistance by $S_U^{E}$. We will also use a superscript $X$ to indicate both cases. The explicit expressions for $S_U$ and $S_U^{E}$ are given in Appendix A. The communication rates for the bidirectional and unidirectional cases are then defined by

$$C_X^* = \sup_{(R_-, R_-) \in S_U^{X}} (R_++R_-),$$

$$C_X^* = \sup_{(R_-, R_-) \in S_U^{X}} R_-, \quad C_X^* = \sup_{(R_-, R_-) \in S_U^{X}} R_-.$$  

Here we have used the $X$ superscript to indicate that the same definitions hold with entanglement assistance and without entanglement assistance.

III. LIMITING THE ANCILLA

In the above definitions, the size of the ancilla is always finite, but has no upper limit, and can grow arbitrarily quickly with $t$. This unbounded nature of the dimension of
the ancilla causes difficulties in deriving results for the communication capacities. To avoid these problems, in this section we show how to put an upper bound on the dimension of the ancilla; that is, we show that for all achievable rate pairs (that are not on the boundary of the region of achievable rate pairs), there exists a constant $K$ such that this rate pair may be achieved with an ancilla space of dimension no larger than $2^K$.

In Ref. [16] similar results were derived for communication without prior entanglement and for one-way communication with entanglement. Here we give a general derivation that may be applied to these cases, and also to bidirectional communication with entanglement.

The set of achievable rate pairs $(R_- , R_+)$, $S_{\rho}^{X}$, forms a two-dimensional region with an upper boundary [16]. The interior of $S_{\rho}^{X}$ is an open set; therefore, for any rate pair $(R_- , R_+)$ in the interior of $S_{\rho}^{X}$, there must be an achievable pair $(R_- ^\prime , R_+ ^\prime)$ closer to the upper boundary, so $R_+ ^\prime > R_+$ and $R_- ^\prime > R_-$. These rate pairs may be for the entanglement assisted or unassisted case; that is, our derivation applies to both cases.

It is possible to achieve the rate pair $(R_- ^\prime , R_+ ^\prime)$ with fidelity $1 - \epsilon$, where
\begin{equation}
\epsilon \leq \min \left( \frac{\Delta R^2}{32 R_{\max}^2} , \frac{\Delta R^4}{16} , \frac{1}{2e} \right),
\end{equation}
with \( \Delta R = \min \{ R_+ ^\prime - R_+ , R_- ^\prime - R_- \} \).

This will have fidelity at least $1 - \epsilon$ with $|x\rangle_{B_1}$. When density matrix $\rho_i$ is given with probability $p_i$, we denote this by the ensemble $E = \{ p_i , \rho_i \}$. From Refs. [19,20], the communication that may be performed with such an ensemble is given by the Holevo information
\begin{equation}
\chi(E) = S \left( \sum_i p_i \rho_i \right) - \sum_i p_i S(\rho_i).
\end{equation}

This communication may be obtained asymptotically by using error correcting codes over multiple states.

For the communication scheme above, the ensemble possessed by Alice is $E_A = \{ 2^{-n_h} \rho_{A_1}^{(y)} \}$. The elements of this ensemble have fidelity at least $1 - \epsilon$ with the ensemble $E_A^0 = \{ 2^{-n_h} |y\rangle A_1 \langle y | \}$. Using the concavity of the fidelity, the average density matrix $\bar{\rho}_{A_1} = 2^{-n_h} \sum \rho_{A_1}^{(y)}$ has fidelity at least $1 - \epsilon$ with the average density matrix for $E_A^0$ given by $\bar{\sigma}_{A_1} = 2^{-n_h} \sum |y\rangle A_1 \langle y |$.

It is easily shown that the Holevo information for $E_A^0$ is $n_b$. To estimate the Holevo information for $E_A$, we may use Fannes’ inequality [21]
\begin{equation}
|S(\rho) - S(\sigma)| \leq T(\rho, \sigma) \log D + \eta(T(\rho, \sigma)),
\end{equation}
where $D$ is the dimension of the Hilbert space and $\eta(x) = -x \log x$. $T(\rho, \sigma) = \| \rho - \sigma \|$ is the trace distance, and we require that $T(\rho, \sigma) \leq 1/\epsilon$.

For $\rho = \rho_{A_1}^{(y)}$ and $\sigma = |y\rangle A_1 \langle y |$, or $\rho = \bar{\rho}_{A_1}$ and $\sigma = \bar{\sigma}_{A_1}$, the fidelity is at least $1 - \epsilon$. Using the inequality between trace distance and fidelity given in Ref. [18], this implies that $T(\rho, \sigma) \leq 2\sqrt{\epsilon}$. The fact that we have taken $\epsilon \leq (1/2e)^2$ implies that $T(\rho, \sigma) \leq 1/\epsilon$ is satisfied.

Using these results, and the fact that $\eta(x) \leq 2x$, the Holevo information for $E_A$ must satisfy
\begin{equation}
\chi(E_A) \approx n_b - 4n \sqrt{\epsilon} - 4e^{1/4},
\end{equation}
where $n = \max\{ n_a , n_b \}$. Using the inequalities $n \leq 2 R_{\max}$, $n_b \geq R_- ^\prime + R_- , \tau \geq 1$ and the restrictions put on $\epsilon$ in Eq. (8), we get
\begin{equation}
\chi(E_A) \geq \tau (R_- ^\prime + R_- )/2.
\end{equation}

From Ref. [19], by coding over a number of ensembles $M$ no less than some minimum $M_\lambda$, it is possible to communicating information per ensemble from Bob to Alice arbitrarily close to $\chi(E_A)$ with average fidelity arbitrarily close to 1. For example, we may achieve communication $M \tau (R_- ^\prime + 3R_- )/4$ with average fidelity $1 - \frac{1}{2} \epsilon_0 (R_- ^\prime + 3R_- )/(R_- ^\prime + 3R_- )$. The fact that the result of Ref. [19] is given in terms of the average fidelity, rather than the minimum fidelity, is unimportant because the codewords that result in low fidelity may be omitted. For this example, no more than a proportion $(R_- ^\prime + 3R_- )/(R_- ^\prime + 3R_- )$ of the codewords may give fidelity less than $1 - \epsilon_0/2$. Omitting these then gives communication of $M \tau R_-$ with fidelity of at least $1 - \epsilon_0/2$.

The same considerations hold for communication from Alice to Bob. For Bob’s ensemble we find
\begin{equation}
\chi(E_B) \geq \tau (R_- ^\prime + R_- )/2.
\end{equation}
Similarly, by coding over a large number \( M \geq M_B \) of ensembles, it is possible to achieve communication from Alice to Bob of \( M \tau R_{-} \) with fidelity of at least \( 1 - \epsilon_2/2 \). If we take the number of ensembles \( M \) to be larger than both \( M_A \) and \( M_B \), then it is possible to perform the coding for communication in each direction on the same block of \( M \) states. In this way, communication \( M \tau R_{-} \) from Bob to Alice and \( M \tau R_{-} \) from Alice to Bob may be performed, with the total fidelity at least \( 1 - \epsilon_0 \).

Thus we see that, by coding over \( M \) blocks of \( \tau \) applications of \( U \), we may perform communication per operation of \( R_{-} \) from Bob to Alice and \( R_{-} \) from Alice to Bob with arbitrarily high fidelity. \( M \) copies of the individual ancillas \( |\psi\rangle_{A_1B_2} \) are required, each of dimension \( d \), for a total dimension of \( d^M \). In addition, we should take account of the \( M \) copies of subsystems \( A_1 \) and \( B_1 \). These subsystems contain the codewords, and the decoded messages will be contained in separate subsystems. We therefore find that the dimension of the ancilla required for the total communication scheme is no more than \( 2^{K_t} \), where \( t = M \tau \) is the total number of operations and \( K = (2n + \log d)/\tau \).

Thus we see that any achievable rate pair \( (R_{-},R_{-}) \) in the interior of \( S_U \) may be achieved with a bound \( 2^{K_t} \) on the dimension of the ancilla. The restriction that the rate pair must be in the interior of \( S_U \) does not affect the results for the communication capacities, because these capacities are defined via a supremum, and we may take the rate pair to be arbitrarily close to the upper boundary.

Note also that, because we have taken \( n_a \) and \( n_b \) for the individual communication schemes (with \( \tau \) applications of \( U \)) to be no larger than \( 2\tau R_{\max} \), the total communication in either direction may be bounded by \( tK_a \), where \( K_a = 2R_{\max} \). This bound is relatively trivial, but will be used later. Another application of these results is that \( t \) may be taken to approach infinity in the limit \( \epsilon \rightarrow 0 \), which is not explicitly given in the definition of the rates.

**IV. COMMUNICATION WITH ENTANGLEMENT**

The fact that it is possible to limit the dimension of the ancilla in this way considerably simplifies some of the analysis in Ref. [16], and allows additional relations to be derived. It is possible to derive inequalities between the classical communication capacity and the entanglement capability using a quantum superposition of classical messages. However, there is the complication that the entanglement only converges if there is an upper limit placed on the dimension.

In Ref. [16] the inequalities \( E_U \geq C_+ \) and \( E_U + E_U \geq C_E \) were proven. The ancillas were bounded using techniques that are specific to these inequalities and cannot be generalized. The bound that we have placed on the dimension of the ancilla in Sec. III is sufficiently general to be applied to derive these inequalities and show further inequalities. In Sec. IV A below, we show the additional inequality \( E_U + E_U \geq C_E \). This inequality is for general bipartite unitary operations, and is not restricted to two-qubit unitaries. In Sec. IV B we show a simplified inequality that is restricted to two-qubit unitary operations.

**A. Arbitrary bipartite unitary operations**

Let \( (R_{-},R_{-}) \) be a pair of rates in the interior of \( S_U \). From Sec. II, there exists a protocol that transmits \( n_a \geq tR_{-} \) bits from Alice to Bob and \( n_b \geq tR_{-} \) bits from Bob to Alice, via \( t \) uses of \( U \) and with fidelity \( 1 - \epsilon \). From Sec. III, there exists a constant \( K \) such that \( \epsilon \) may be made arbitrarily small, while the dimension of the ancilla is no greater than \( 2^{K_t} \). The final state \( |\eta_{xy}\rangle_{AB} \) satisfies

\[
F(|\eta_{xy}\rangle_{AB}, |\eta_{xy}\rangle_{AB}) = 1 - \epsilon_{xy},
\]

with \( \epsilon_{xy} \leq \epsilon \) for all messages \( x \) and \( y \). Using Uhlmann’s theorem, the state \( |\eta_{xy}\rangle_{AB} \) may be expressed as [22]

\[
|\eta_{xy}\rangle_{AB} = \sqrt{1 - \epsilon_{xy}}|\eta_{xy}\rangle_{AB} + \epsilon_{xy}|\epsilon_{xy}\rangle_{AB},
\]

where \( Tr_{A_2B_2}|\epsilon_{xy}\rangle_{AB} \) has support orthogonal to \( |\eta_{xy}\rangle_{A_1B_1} \). The change in the entanglement is

\[
\Delta E_{xy} = E(|\eta_{xy}\rangle_{AB}) - E(|\eta_{xy}\rangle_{AB}).
\]

Now we add additional subsystems \( A_3 \) and \( B_3 \) that contain copies of Alice’s message \( x \) and Bob’s message \( y \). In addition, rather than selecting specific classical messages \( x \) and \( y \), we have a coherent superposition of all possible messages. Then the input state is

\[
2^{-\left(n_a+n_b/2\right)} \sum_{xy} |\eta_{xy}\rangle_{A_1B_1} |\eta_{xy}\rangle_{A_1B_1} |\eta_{xy}\rangle_{A_2B_2},
\]

and the output state is

\[
|\eta\rangle = 2^{-\left(n_a+n_b/2\right)} \sum_{xy} |\eta_{xy}\rangle_{A_1B_1} |\eta_{xy}\rangle_{A_2B_2}.
\]

Here we use the notation of multiple subscripts to indicate the combined Hilbert space, for example, \( A_{12} \) indicates \( \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \). We use \( A \) and \( B \) to indicate the entire Hilbert spaces of Alice and Bob, respectively; so, for example, \( A \) is equivalent to \( A_{12} \).

In order to put bounds on the entanglement that may be created, we compare Eq. (21) with the state

\[
|\eta\rangle = 2^{-\left(n_a+n_b/2\right)} \sum_{xy} |\eta_{xy}\rangle_{A_1B_1} |\eta_{xy}\rangle_{A_2B_2}.
\]

We find that

\[
\langle \eta| \eta\rangle = 2^{-\left(n_a+n_b/2\right)} \sum_{xy} \langle \eta_{xy}|_{A_1B_1} \langle \eta_{xy}|_{A_2B_2} \rangle_{xy} \langle \eta_{xy}|_{A_2B_2} \rangle_{12}
\]

\[
\geq 2^{-\left(n_a+n_b/2\right)} \sum_{xy} \sqrt{1-\epsilon} = \sqrt{1-\epsilon},
\]

giving \( \langle \eta| \eta\rangle \geq 1 - \epsilon \).

From the continuity of the entropy of entanglement [23], or Fannes’ inequality [21], we find that the difference between the entanglement of the two states \( |\eta\rangle \) and \( |\eta_e\rangle \) is bounded by

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\[ |E(\eta) - E(\eta_{\phi})| \leq T(\eta, \eta_{\phi})(2n + Kt/2) + Q. \]  

(24)

where \( n = \max(n_a, n_b), \) \( Q = \log(e)/e \) and \( T(|a\rangle, |b\rangle) = \left| \text{Tr}[|a\rangle\langle a| - |b\rangle\langle b|] \right| \) is the trace distance. Here we have used the fact that the dimension of the ancilla is bounded by \( 2^{K_t}. \) Using the inequality between trace distance and fidelity given in Ref. [18], we find that

\[ |E(\eta) - E(\eta_{\phi})| \leq \sqrt{1 - \langle \eta | \eta_{\phi} \rangle^2(4n + Kt)} + Q \leq \sqrt{\epsilon(4n + Kt)} + Q. \]  

(25)

Therefore the increase in the entanglement for \( t \) applications of \( U \) is

\[ \Delta E = E(|\eta\rangle) - E(|\phi_{A_2B_2}\rangle) \]

\[ \geq E(|\eta\rangle) - \sqrt{\epsilon(4n + Kt)} - Q - E(|\phi_{A_2B_2}\rangle). \]  

(26)

The entanglement of \( |\eta\rangle \) may be found, using Eq. (11.58) of Ref. [18], to be

\[ E(|\eta\rangle) = n_a + n_b + 2^{-(n_a + n_b)} \sum_{xy} E(|c_{xy}\rangle_{A_2B_2}). \]  

(27)

Substituting this into Eq. (26) gives

\[ \Delta E \geq n_a + n_b - \sqrt{\epsilon(4n + Kt)} - Q \]

\[ + 2^{-(n_a + n_b)} \sum_{xy} [E(|c_{xy}\rangle_{A_2B_2}) - E(|\phi_{A_2B_2}\rangle)]. \]  

(28)

Again applying the continuity of the entropy of entanglement [23],

\[ |E(|\eta_{AB}\rangle_{A_1B_1}) - E(|\eta_{AB}\rangle_{A_1B_1})| \leq T(|\eta_{AB}\rangle_{A_1B_1}, |\eta_{AB}\rangle_{A_1B_1}) \]

\[ \leq T(|\eta_{AB}\rangle_{A_1B_1}, |\eta_{AB}\rangle_{A_1B_1}) \]

\[ \leq \sqrt{\epsilon(2n + Kt)} + Q. \]  

(29)

Using this, and the fact that \( \Delta E_{\phi} \geq -tE_{U\phi} \), we find

\[ E(|c_{xy}\rangle_{A_2B_2}) - E(|\phi_{A_2B_2}\rangle) \geq E(|\eta_{AB}\rangle_{A_1B_1}) - \sqrt{\epsilon(2n + Kt)} \]

\[ - Q - E(|\phi_{A_2B_2}\rangle) \]

\[ \geq -tE_{U\phi} - \sqrt{\epsilon(2n + Kt)} - Q. \]  

(30)

Substituting this into Eq. (28), and noting that \( \Delta E \approx tE_{U\phi} \), gives

\[ tE_{U\phi} \geq n_a + n_b - tE_{U\phi} - 2\sqrt{\epsilon[3n + Kt]} - 2Q. \]  

(31)

Using the inequalities \( n \geq tK_n \) (from Sec. III), \( n_a \geq tR_\alpha \), \( n_b \geq tR_\beta \), and dividing on both sides by \( t \), gives

\[ E_{U\phi} + E_{U\phi} \geq R_\alpha + R_\beta - 2\sqrt{\epsilon[3K_n + K]} - 2Q/t. \]  

(32)

In the limit \( \epsilon \to 0 \), as \( K \) and \( K_n \) are fixed, the third term on the right-hand side goes to zero. The fourth term goes to zero in this limit, because \( t \) may be taken to approach infinity (as mentioned in the preceding section). Therefore we obtain

\[ E_{U\phi} + E_{U\phi} \geq R_\alpha + R_\beta, \]  

for any achievable pair of rates \( (R_\alpha, R_\beta) \) that is in the interior of \( S_f^E \). This implies that

\[ E_{U\phi} + E_{U\phi} \geq C^E_{\phi}. \]  

(33)

Note that it is the upper limit \( K \) on the number of auxiliary qubits that may be used per operation that allows us to take the limit \( \epsilon \to 0 \). When the dimension of the system is limited in this way, the entanglement has good continuity properties. This continuity means that we obtain the same result in the limit \( \epsilon \to 0 \) as we would if the communication were exact (\( \epsilon = 0 \)).

**B. Two-qubit unitary operations**

For the case of two-qubit unitaries, we may simplify this result. In this case the maximum increase in the entanglement \( E_{U\phi} \) is equal to the maximum decrease in the entanglement \( E_{U\phi} \), which may be shown in the following way. Any two-qubit interaction \( U \) is equivalent, up to local unitary operations, to an operation of the form \([3,5,24,25]\)

\[ U_d(\alpha_1, \alpha_2, \alpha_3) = e^{-i(\alpha_1\sigma_1 \otimes \sigma_1 + \alpha_2\sigma_2 \otimes \sigma_2 + \alpha_3\sigma_3 \otimes \sigma_3)}. \]  

(34)

where \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) are the Pauli sigma matrices.

As entanglement capability and classical communication capacity are independent of local unitary operations, we may restrict to operations of this form. It is then simple to show that \( U_d^* = U_d = U_d^{-1} \). As discussed in Ref. [5], for any measure of entanglement, \( E(|\Psi\rangle) = E(|\Psi^*\rangle) \). This means that, if the operation \( U_d \) acting on the state \( |\Psi\rangle \) generates the maximum increase in entanglement, then this operation performed on the state \( U_d^*|\Psi\rangle \) decreases the entanglement by \( E_{U\phi} \). Therefore, the operation may decrease the entanglement at least as much as it may increase it. Similarly, it is simple to show the converse, and therefore \( E_{U\phi} = E_{U\phi} \). Thus we find that, for two-qubit unitary operations, we have the inequality

\[ 2E_{U\phi} = C^E_{\phi}. \]  

(35)

**V. COMMUNICATION WITHOUT ENTANGLEMENT**

For the case of two-qubit unitaries we may obtain inequalities between the communication that may be obtained with and without entanglement. To see this, again consider any pair of rates \( (R_\alpha, R_\beta) \) in the interior of \( S_f^E \). We may select a second pair \( (R_\alpha', R_\beta') \) that is closer to but not on the upper boundary, so that \( R_\alpha' > R_\alpha \) and \( R_\beta' > R_\beta \).

Then there exists a communication scheme that communicates \( n_{\alpha'} \approx \tau R_\alpha' \) bits from Alice to Bob and \( n_{\beta'} \approx \tau R_\beta' \) bits from Bob to Alice via the uses of \( U \) and with fidelity \( 1 - \epsilon \). The input state is \( |x\rangle_{A_1}|y\rangle_{B_1}|\phi_{A_2B_2}\rangle \) and the output state \( |\eta_{AB}\rangle_{A_2B_2} \) is obtained via a process of the form of Eq. (4).

The process described by Eq. (4) consumes entanglement resources to perform this communication. We consider a strategy for recovering the initial entanglement resource state
by subsequently performing the communication scheme (4) in reverse. (This approach is analogous to that applied in Ref. [26].) To achieve this, Alice and Bob must again retain copies of their inputs in auxiliary subsystems $A_4$ and $B_4$. In addition, to ensure that at the end of this process Alice and Bob retain the communicated information, Bob must create a copy of the output $x$ and Alice must create a copy of $y$. We therefore add auxiliary subsystems $A_4$ and $B_4$ that are initially in the state $|0\rangle$. After performing the communication we copy the value $y$ from $A_1$ to $A_4$ and copy the value $x$ from $B_1$ to $B_4$.

From Eq. (18), we may express the state after the communication process, but before the outputs are copied into $A_4$ and $B_4$, as

$$\{ \sqrt{1-e_{xy}} |y\rangle_{A_1} |x\rangle_{B_1} e_{xy} A_{2} B_{2} + \sqrt{e_{xy}} e_{xy} A_{1} B_{12} \} \otimes |x\rangle_{A_3} |y\rangle_{B_3} |0\rangle_{A_4} |0\rangle_{B_4},$$

(36)

where $|e_{xy}\rangle_{A_{12}B_{12}}$ is some normalized error state. Copying the results of the communication to the auxiliary subsystems $A_4$ and $B_4$ then yields

$$\sqrt{1-e_{xy}} |y\rangle_{A_1} |x\rangle_{B_1} e_{xy} A_{2} B_{2} |y\rangle_{A_4} |x\rangle_{B_4} A_{3} A_3 |y\rangle_{B_3} + \sqrt{e_{xy}} e_{xy} A_{12} B_{12} |x\rangle_{A_1} |y\rangle_{B_1}.$$

(37)

Here we are using superscripts on the error state to indicate that it has also been changed under this operation.

In order to reverse the communication scheme, we wish to apply the inverse of the sequence of operations that was used to perform the communication:

$$(V_{A_{12}}^{0}) \otimes V_{B_{12}}^{(0)} U^{U^I} \cdots U^{U^I} (V_{A_{12}}^{(r)} \otimes V_{B_{12}}^{(r)})^{T}.$$  

(38)

It is possible to apply the inverses of the local operations $V_{A_{12}}^{(r)}$ and $V_{B_{12}}^{(r)}$, because we assume that it is possible to perform arbitrary local unitary operations. Applying the inverse of operation $U$ is more difficult, however. When the operation $U$ is in the simple form (34), the inverse operation $U^I$ is equal to $U^*$. If we take the composite conjugate of the state, then the action of $U$ on this complex conjugate state will be the same as that of $U^*$ on the original state. To reverse the communication scheme on the composite conjugate state, we simply apply the complex conjugate of the sequence of operations in Eq. (38):

$$(V_{A_{12}}^{0}) \otimes V_{B_{12}}^{(0)} U^{T} \cdots U^{T} (V_{A_{12}}^{(r)} \otimes V_{B_{12}}^{(r)})^{T}.$$  

(39)

Here we can not take the exact complex conjugate of the state, but we can take the complex conjugate of the first term in Eq. (37). To obtain the complex conjugate of this term, note that we may express $|c_{xy}\rangle_{A_2 B_2}$ via a Schmidt decomposition, in the form

$$|c_{xy}\rangle_{A_2 B_2} = \sum_{i} \sqrt{\lambda_{xy}^{i}} |\phi_{xy}^{i}\rangle_{A_2} |\chi_{xy}^{i}\rangle_{B_2}.$$  

(40)

The first term in Eq. (37) is therefore proportional to the state

$$|y\rangle_{A_1} |x\rangle_{A_4} |y\rangle_{B_1} |x\rangle_{B_4} \sum_{i} \sqrt{\lambda_{xy}^{i}} |\phi_{xy}^{i}\rangle_{A_2} |\chi_{xy}^{i}\rangle_{B_2}.$$  

(41)

We may take the complex conjugate of this state using the local operations

$$\sum_{xy} |y\rangle_{A_1} |y\rangle_{B_1} \otimes |x\rangle_{A_3} |x\rangle_{A_4} \otimes \sum_{i} |\phi_{xy}^{i}\rangle_{A_2} |\phi_{xy}^{i}\rangle_{B_2},$$  

(42)

$$\sum_{xy} |y\rangle_{B_1} |y\rangle_{B_3} \otimes |x\rangle_{B_4} |x\rangle_{B_4} \otimes \sum_{i} |\chi_{xy}^{i}\rangle_{B_2} |\chi_{xy}^{i}\rangle_{B_2}.$$  

(43)

These are conditional operations with $A_1$ and $A_3$ (for Alice) and $B_1$ and $B_3$ (for Bob) as the controls. Note that it is retaining the value of $x$ in $A_3$ and the value of $y$ in $B_3$ that makes these operations possible.

Applying these operations gives the state

$$\sqrt{1-e_{xy}} |y\rangle_{A_1} |x\rangle_{B_1} e_{xy} A_{2} B_{2} |y\rangle_{A_4} |x\rangle_{B_4} A_{3} A_3 |y\rangle_{B_3} + \sqrt{e_{xy}} e_{xy} A_{12} B_{12} |x\rangle_{A_1} |y\rangle_{B_1}.$$  

(44)

Now we have a state that is close to the complex conjugate of the output state of the communication process. It is therefore possible to approximately reverse the communication process by performing the sequence of operations given in Eq. (39) to state (44). In this way, via $2 \tau$ implementations of $U$, we obtain the output state

$$\{( |x\rangle_{A_1} |y\rangle_{B_1} |\phi^{*}\rangle_{A_2 B_2} - \sqrt{e_{xy}} e_{xy} A_{12} B_{12} |y\rangle_{A_4} |x\rangle_{B_4} \} + \sqrt{e_{xy}} e_{xy} A_{12} B_{12} |x\rangle_{A_1} |y\rangle_{B_1}.$$  

(45)

Then we perform local transformations that take the complex conjugate of $|\phi^{*}\rangle_{A_2 B_2}$, giving the final state

$$|\zeta_1\rangle = |x\rangle_{A_1} |x\rangle_{A_4} |y\rangle_{B_1} |y\rangle_{B_4} |\psi_{A_2 B_2}| |y\rangle_{A_4} |x\rangle_{B_4}$$  

$$- \sqrt{e_{xy}} e_{xy} A_{12} B_{12} |y\rangle_{A_4} |x\rangle_{B_4}.$$  

(46)

Here we have omitted the subscript $AB$ on states in the entire Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ for brevity.

We now consider a sequence of $M$ processes of this type, communicating $M$ messages $x_1$ to $x_M$ from Alice to Bob and $M$ messages $y_1$ to $y_M$ from Bob to Alice. If we use the output state from the first communication process (46), and perform the same communication process to communicate the second two messages $x_2$ and $y_2$, then the state we will obtain is

$$|\zeta_2\rangle = |x_1 x_2\rangle_{A_1} |x_1 x_2\rangle_{A_4} |y_1 y_2\rangle_{B_1} |y_1 y_2\rangle_{B_4} |\phi_{A_2 B_2}| |y_1 y_2\rangle_{A_4} |x_1 x_2\rangle_{B_4}$$  

$$- \sqrt{e_{xy}} e_{xy} A_{12} B_{12} |y_1 y_2\rangle_{A_4} |x_1 x_2\rangle_{B_4} - \sqrt{e_{xy}} e_{xy} A_{12} B_{12} |x_1 x_2\rangle_{A_4} |y_1 y_2\rangle_{B_4}$$  

$$- \sqrt{e_{xy}} e_{xy} A_{12} B_{12} |y_1 y_2\rangle_{B_4} + \sqrt{e_{xy}} e_{xy} A_{12} B_{12} |x_1 x_2\rangle_{B_4}.$$  

(47)

Doing this $M$ times, each step generates an additional two error terms of the form given in Eq. (46), resulting in the final state
Without prior entanglement, we construct an approximation \( K \) than means that the entanglement of this state cannot be larger to the state for perfect communication satisfies \( C \). Here we have used \( \epsilon \), dimension of \( U \) required is no more than \( K \). If the initial state \( u \) be a state with entanglement \( E_0 \) and fidelity at least 1 - \( \epsilon \) provided we create the initial state with fidelity 1 - \( \epsilon \) and initial entanglement required by selecting the number of repetitions \( M \) to be sufficiently large so that inequalities (53) are satisfied. For this value of \( M \) we ensure that the fidelity 1 - \( \epsilon \) is achieved by selecting the block size \( \tau \) to be sufficiently large so that the inequality \( 4M \sqrt{\epsilon} = \epsilon \) is satisfied.

Thus, for two-qubit unitary operations, if the rate pair \( (R_+, R_-) \) is in the interior of \( J^U \), the rate pair \( (R_+/2, R_-/2) \) is achievable without entanglement assistance. As \( R_+ - R_- \) may be taken to be arbitrarily close to \( C^E \), we have the inequality for two-qubit unitaries \( C^E = 2C_+ \).

B HLS derive the inequality \( C_+ = E_U \), giving the ordering of the capacities for two-qubit unitary operations \( C^E = 2C_+ = 2E_U \).

Similarly, \( R_+ \) may be taken to be arbitrarily close to \( C^E \), and \( R_- \) may be taken to be arbitrarily close to \( C^E \), implying the further inequalities for two-qubit unitary operations \( C_+^E \leq 2C_+ \) and \( C_-^E = 2C_- \). For two-qubit unitaries the communication capacities in each direction are identical [16], so these inequalities may alternatively be expressed as \( C_+^E = C_-^E \leq 2C_+ = 2C_- \).

Note that these results are crucially dependent on the fact that we have limited the dimension of the ancilla that may be used. The number of operations required to create the initial state scales no faster than \( O(K \tau) \), and \( O(K \tau) /M \tau \) approaches zero in the limit \( M \to \infty \). If the number of operations required to create the initial state scaled superlinearly with \( \tau \), then it would not be possible to take this limit.

VI. Bidirectional Ensembles

A powerful means of deriving results for the case of communication in a single direction is by considering the Holevo information of an ensemble of states. When a joint state shared between Alice and Bob, \( |\psi_i\rangle \), is given with probability \( p_i \), we denote this as the ensemble \( E = \{p_i, |\psi_i\rangle \} \). Similarly, we may denote the ensemble of reduced density matrices possessed by Bob by \( E = \{p_i, \rho_i\} \), where \( \rho_i = \text{Tr}_A(\psi_i \langle \psi_i |) \). The index \( i \) is chosen by Alice, and may be used to communicate classical information. If a long se-
quence of the states $|\psi_i\rangle$ is given with probabilities $p_i$, Alice may perform communication equal to $\chi(E)$ [19,20].

B HLS show that the communication capacity in a single direction may be evaluated by considering the Holevo information before and after applying the operation $U$: that is, we may define

$$
\Delta \chi_\ell = \sup_{E} [\chi(\text{Tr}_A U E) - \chi(\text{Tr}_X E)].
$$

(57)

We take the convention that

$$
U E = \{p_i, U|\Phi_i\rangle\},
$$

(58)

$$
\text{Tr}_X E = \{p_i, \text{Tr}_X(|\Phi_i\rangle)\},
$$

(59)

where $\text{Tr}_X(|\psi\rangle) = \text{Tr}_X(|\psi\rangle).$ We use the superscript “$\ell$” to indicate that this is the difference in the Holevo information before and after applying the operation $U$.

For communication from Bob to Alice. It is shown in Ref. [16] that $C^E = \Delta \chi_\ell$ and $C_E = \Delta \chi_\ell$.

To apply similar ideas to the case of bidirectional communication. In this case, let $i$ be the message encoded by Alice and $j$ be the message encoded by Bob. We may then define a bidirectional ensemble by

$$
E = \{p_i, q_j, |\psi_{ij}\rangle\}.
$$

(60)

In general, Bob has a reduced density matrix that depends on $i$ and $j$, which he may use to obtain information about $i$. Similarly, Alice has a reduced density matrix dependent on $i$ and $j$, and may perform measurements to gain information about $j$.

In order to apply the same ideas as for the case of communication in a single direction, it is necessary to consider the bidirectional communication that may be performed using this ensemble. Here we give upper and lower bounds, and we will show in Sec. VII B that these bounds coincide for a particular type of ensemble.

In order to obtain an upper bound, we may assume that Alice and Bob use information about the other’s message in their encoding. If Alice knows in advance the values of $j$ that Bob will be using, then she may perform block coding over those ensembles with the same value of $j$. The information that Alice may communicate to Bob for each value of $j$ is given by

$$
S \left[ \sum_i p_i \rho^B_{ij} - \sum_i p_i S(\rho^B_{ij}) \right].
$$

(62)

where $\rho^B_{ij} = \text{Tr}_A|\psi_{ij}\rangle$. Averaging over the values of $j$, the average communication is equal to

$$
\chi_{\text{up}}(E) = \sum_j q_j S \left[ \sum_i p_i \rho^B_{ij} - \sum_i p_i S(\rho^B_{ij}) \right].
$$

(63)

When Alice does not have advance information about the values of $j$, she will not be able to perform the coding in this way. Nevertheless, it is obvious that it will not be possible to perform more communication without prior information about the values of $j$. Therefore $\chi_{\text{up}}(E)$ gives an upper limit to the communication that may be performed from Alice to Bob using this ensemble. In exactly the same way, we see that an upper limit to the communication possible from Bob to Alice is given by

$$
\chi_{\text{up}}(E) = \sum_j p_j S \left[ \sum_i q_i \rho^A_{ij} - \sum_i q_i S(\rho^A_{ij}) \right],
$$

(64)

where $\rho^A_{ij} = \text{Tr}_B|\psi_{ij}\rangle$.

We may obtain a lower bound by assuming that Alice and Bob code independently of the messages being encoded by the other party. If Alice chooses message $i$, and has no information about $j$, she may take Bob’s density matrix to be $\rho^A_{ij} = \sum_j q_j \rho^B_{ij}$.

In order to improve upon this, Bob may apply some $j$-dependent completely positive map $T^B_j$ to his portion of the state. The idea behind using this map is to reduce the dependence of Bob’s reduced density matrix, so the averaged density matrices $\rho^A_{ij}$ are more distinguishable for different values of $i$. With this operation, Alice may take Bob’s density matrix to be

$$
\rho^B_{ij} = \sum_j q_j T^B_j (\rho^B_{ij}).
$$

(65)

Via coding over multiple ensembles, Alice may perform communication to Bob equal to the Holevo information of the ensemble $\{p_i, \rho^B_{ij}\}$. We now take the supremum of this over the maps $\{T^B_j\}$, and define

$$
\chi_{\text{lo}}(E) = \sup \chi_{\text{up}}(E) = \chi_{\text{up}}(E),
$$

(66)

where $\rho^B_{ij}$ is defined as in Eq. (65). This is a lower limit to the communication that may be performed from Alice to Bob. Similarly a lower limit to the communication that may be performed from Bob to Alice is

$$
\chi_{\text{lo}}(E) = \sup \chi_{\text{up}}(E) = \chi_{\text{up}}(E),
$$

(67)

where

$$
\rho^A_{ij} = \sum_j p_j T^A_j (\rho^A_{ij}),
$$

(68)

and $T^A_j$ is a completely positive map applied by Alice.

We will also denote the exact values of the maximum communication that it is possible to communicate from Alice to Bob and from Bob to Alice by $\chi^{-}(E)$ and $\chi^{-}(E)$, respectively. From the definitions, it is evident that

$$
\chi_{\text{lo}}(E) = \chi^{-}(E) \leq \chi_{\text{up}}(E), \quad \chi_{\text{lo}}(E) = \chi^{-}(E) \leq \chi_{\text{up}}(E).
$$

(69)

We may also consider the maximum bidirectional communication that it is possible to perform with the ensemble, $\chi^{-}(E) = \chi^{-}(E) + \chi^{-}(E)$. This quantity has the upper and lower bounds
\[ x_{i0}^\perp(E) = x_{i0}^\perp + x_{i0}^\perp, \quad x_{i0}^\perp(E) = x_{i0}^\perp + x_{i0}^\perp. \]  

(70)

Analogous to the case for communication in a single direction, we may define the maximum that the classical communication capacity of an ensemble may be increased,

\[ \Delta x_u^\perp = \sup_{E} [x^\perp(U,E) - x^\perp(E)]. \]  

(71)

This quantity is analogous to the unidirectional quantity \( \Delta x_u^\perp \) introduced by BHLS. However, in contrast to the unidirectional case, it has not been proven that \( C_u^E = \Delta x_u^\perp \).

**VII. ENSEMBLES FOR ENTANGLEMENT-ASSISTED COMMUNICATION**

In this section we show how to obtain initial ensembles for two-qubit operations such that the communication possible in one or both directions may be increased by \( E_U \). We first consider the simpler case of communication in a single direction, then extend this to the bidirectional case.

**A. Communication in a single direction**

We consider the case of \( \Delta x_u^\perp \). As all two-qubit operations are symmetric, the results for this case also apply to \( \Delta x_u^\perp \). Note that the second term on the right-hand side of Eq. (12) is the average of the entanglement of the coding states. Therefore, if each of the states of the initial ensemble are chosen such that the entanglement of these states is decreased by the maximum \( E_U \) by operation \( U \), then the second term in Eq. (12) will be decreased by \( E_U \) by the operation. If the first term is constant, then the total increase in the Holevo information will be \( E_U \).

In order to obtain such an ensemble, let us start with an initial state \( |\Psi\rangle \) such that the entanglement is decreased by the maximum \( E_U \) via operation \( U \). (Recall that \( E_U = E_U \) for two-qubit operations.) We then obtain the coding states via a set of operations \( \{V_i \otimes V_i^\perp\} \), where \( V_i \) and \( V_i^\perp \) are local operations on Alice’s and Bob’s sides, respectively. We use the notation where (1) indicates Alice’s ancilla, (2) indicates Alice’s qubit upon which \( U \) acts, (3) indicates Bob’s qubit upon which \( U \) acts, and (4) indicates Bob’s ancilla. The ancillas may be qubits or arbitrary dimensional qudits. In addition, we use \( A \) to indicate Alice’s entire system (1 and 2), and we use \( B \) to indicate Bob’s entire system (3 and 4). In this section we indicate the subsystem with superscripts (rather than subscripts) to avoid notational conflicts.

The operations \( V_i = V_i^A \otimes V_i^B \) must satisfy two criteria.

(1) \( V_i \) commutes with \( U \). This means that each state \( V_i |\Psi\rangle \) will have its entanglement decreased by \( E_U \) for operation \( U \).

(2) For all states \( |\phi\rangle \),

\[ \sum_i p_i \text{Tr}_A(V_i |\phi\rangle) = \frac{1}{2} \otimes \rho^{(4)}. \]  

(72)

where \( \rho^{(4)} = \text{Tr}_{A3}(|\phi\rangle) \) is the reduced density matrix for Bob’s ancilla (4).

As the operation \( U \) does not act on the ancilla, \( \rho^{(4)} \) will be unchanged under \( U \). That is,

\[ \sum_i p_i \text{Tr}_A(V_i |\Psi\rangle) = \sum_i p_i \text{Tr}_A(V_i U |\Psi\rangle) = \frac{1}{2} \otimes \rho^{(4)}. \]  

(73)

This means that the first term on the right side of Eq. (12) will be unchanged by the operation \( U \), and therefore that the total one-way communication possible is \( E_U \).

For \( U \) in the form of Eq. (34), the above conditions are satisfied for the operators \( V_i = \sigma_i^{(2)} \otimes \sigma_i^{(3)} \), for \( i \in \{0,1,2,3\} \). Here \( \sigma_i \) are the Pauli matrices for \( i \in \{1,2,3\} \), \( \sigma_0 \) is the identity, and the superscripts indicate the qubits upon which the operators act. It is easy to see that these operators commute with \( U \) in the form of Eq. (34). To show the second property, let us express the reduced density matrix at Bob’s side, \( \rho^{B} = \text{Tr}_A(|\Psi\rangle) \), in the form

\[ \rho^{B} = \frac{1}{2} \sum_{k=0}^{3} \sigma_k^{(3)} \otimes \rho_k^{(4)}. \]  

(74)

It is easily seen that, taking \( p_i = 1/4 \), we obtain

\[ \sum_i p_i \text{Tr}_A(V_i |\Psi\rangle) = \sum_i p_i \sigma_i^{(3)} \otimes \rho_k^{(4)} = \frac{1}{2} \sigma_i^{(3)} \otimes \rho_0^{(4)}. \]  

(75)

Since the trace over qubit \( 3 \) gives \( \rho^{(4)} \), and the trace of each of \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) is equal to zero, we must have \( \rho_0^{(4)} = \rho^{(4)} \). We therefore obtain Eq. (72), satisfying the second condition for the operations \( V_i \). Thus we find that, by coding with equal probabilities \( p_i = 1/4 \) each of the four states \( \sigma_i^{(2)} \sigma_i^{(3)} |\Psi\rangle \), it is possible to increase the Holevo information by \( E_U \). Thus we have proven the inequality \( \Delta x_u^\perp = E_U \). Using the equality \( C_u^E = \Delta x_u^\perp \) [16], we have the inequality \( C_u^E \leq E_U \).

**B. Bidirectional communication**

Next we apply a similar coding protocol to the more complicated case of bidirectional communication. We start with a state \( |\Psi\rangle \) such that \( U \) decreases the entanglement by the maximum \( E_U \). Alice encodes via the set of four operators \( \{\sigma_i^{(2)} \sigma_i^{(3)}\} \), and Bob encodes via the four operators \( \{\sigma_i^{(2)} \sigma_i^{(3)}\} \). Alice and Bob’s operators commute, so the order in which these operators are applied is irrelevant. We therefore have a total ensemble of 16 states

\[ E = \{p_i q_j \sigma_i^{(2)} \sigma_j^{(3)} |\Psi\rangle\} \]  

(76)

where we take \( p_i = q_j = 1/4 \). Note that each of the operators applied by Alice and Bob commutes with \( U \), for \( U \) in the form of Eq. (34). Therefore, for both the initial and final ensembles, the members of the ensemble are of the form \( |\psi_{ij}\rangle = \sigma_i^{(2)} \sigma_j^{(3)} |\phi\rangle \), where \( |\phi\rangle = |\Psi\rangle \) for the initial ensemble and \( |\phi\rangle = U |\Psi\rangle \) for the final ensemble. This simple form allows us to determine the exact bidirectional communication that is possible to perform.
We now take the completely positive maps $T_i^A$ and $T_j^B$ to be the simple unitary operations
\begin{equation}
T_i^A(\rho) = \sigma_i^{(2)} \rho \sigma_i^{(2)}, \quad T_j^B(\rho) = \sigma_j^{(3)} \rho \sigma_j^{(3)}.
\end{equation}
This gives
\begin{equation}
\rho_i^A = \sum_j q_j T_j^B[\sigma_i^{(3)} \rho \sigma_i^{(3)} \operatorname{Tr}_A(|\phi\rangle) \sigma_i^{(3)}] = \sum_j q_j [\sigma_i^{(3)} \operatorname{Tr}_A(|\phi\rangle) \sigma_i^{(3)}] = \sigma_i^{(3)} \operatorname{Tr}_A(|\phi\rangle) \sigma_i^{(3)},
\end{equation}
and similarly
\begin{equation}
\rho_j^B = \sum_i p_i T_i^A[\sigma_j^{(2)} \rho \sigma_j^{(2)} \operatorname{Tr}_B(|\phi\rangle) \sigma_j^{(2)}] = \sigma_j^{(2)} \operatorname{Tr}_B(|\phi\rangle) \sigma_j^{(2)}.
\end{equation}
We then obtain the Holevo information of the ensembles $E^A = \{p_i, \rho_i^A\}$ and $E^B = \{q_j, \rho_j^B\}$ as
\begin{equation}
\chi(E^A) = S\left(\frac{1}{2} \otimes \rho^{(4)}\right) - S[\operatorname{Tr}_A(|\phi\rangle)] = \chi_{up}(E),
\end{equation}
\begin{equation}
\chi(E^B) = S\left(\frac{1}{2} \rho^{(1)} \otimes 1\right) - S[\operatorname{Tr}_B(|\phi\rangle)] = \chi_{up}(E).
\end{equation}
Here we have used
\begin{equation}
\sum_i \operatorname{Tr}_A(\sigma_i^{(2)} \sigma_i^{(3)} \sigma_j^{(2)} \sigma_j^{(3)} |\phi\rangle) \otimes 1 = \rho^{(4)},
\end{equation}
\begin{equation}
\sum_j \operatorname{Tr}_B(\sigma_i^{(2)} \sigma_i^{(3)} \sigma_j^{(2)} \sigma_j^{(3)} |\phi\rangle) = \rho^{(1)} \otimes 1,
\end{equation}
where $\rho^{(1)} = \operatorname{Tr}_B(|\phi\rangle)$ is the reduced density matrix for Alice’s ancilla.
As we have taken the lower limits $\chi_{\infty}(E)$ and $\chi_{\infty}(E)$ to be the suprema over completely positive maps of $\chi(E^A)$ and $\chi(E^B)$, and we have inequalities (69), we must have
\begin{equation}
\chi_{\infty}(E) = \chi\left(\chi_{\infty}(E)\right) = \chi\left(\chi_{\infty}(E)\right) = \chi\left(\chi_{\infty}(E)\right).
\end{equation}
This implies that $\chi_{\infty}(E) = \chi_{up}(E) = \chi_{\infty}(E)$.
Therefore, for this specific type of bidirectional ensemble, the upper and lower limits on the bidirectional communication that it is possible to perform coincide, and we have an explicit expression for the bidirectional communication that may be performed. To understand the reason for this, Bob performs local operations that remove the $j$ dependence of his part of the shared entangled state. This allows Alice to perform coding without regard for the value of $j$. Similarly, Bob may perform coding without regard for the value of $i$.
Now note that, under the action of the operation $U$, the first terms in Eqs. (80) and (81) are constant (because the operation does not act upon the ancillas). Also the entangle-
ment $S[\operatorname{Tr}_A(|\phi\rangle)] = S[\operatorname{Tr}_B(|\phi\rangle)]$ decreases by $U_E$. Therefore the total increase in the bidirectional communication possible is equal to $2E_U$.
As we have given an explicit scheme that increases the bidirectional communication that it is possible to perform by $2E_U$, we have proven the inequality for two-qubit operations $2E_U \leq \Delta \chi^\rightarrow_U$. Using the other inequalities derived in the preceding sections, we have the sequence of inequalities $C^E_\rightarrow \leq 2C^E_\rightarrow \leq 2E_U \leq \Delta \chi^\rightarrow_U$.

For the cases of the CNOT and SWAP operations, these inequalities are equalities. A potential way of proving equalities in the general case is to show that ensembles of the form of Eq. (76) can be prepared using average bidirectional communication of $\chi\left(\chi\right)$ per ensemble. Such preparation is efficient in the sense that the communication required to prepare the ensembles is the same as what may be performed using the ensembles.

In the unidirectional case, it is known that ensembles can be prepared efficiently [28,29]. BHLS apply this result to show that, given an ensemble $E$ such that the Holevo information is increased by $\Delta \chi^\rightarrow = \chi(\operatorname{Tr}_U E) - \chi(\operatorname{Tr}_U E)$, the entanglement-assisted communication capacity in a single direction is at least $\Delta \chi$. This proof is based upon a communication scheme where initial communication is obtained by some finite number of operations, and used to create initial ensembles. The Holevo information of these ensembles is increased via application of the operation $U$ and used to obtain further communication. This communication may be used to obtain further initial ensembles, and the process is repeated. This process may be repeated a large number of times, so that the average communication per operation is equal to $\Delta \chi$.

It is clear that a similar communication scheme could be applied in the bidirectional case, if it were possible to efficiently create initial ensembles. Not all bidirectional ensembles can be prepared efficiently. For example, consider the ensemble $E = \{1/2, 1/2, |\psi_{ij}\rangle\}$, where $|\psi_{00}\rangle = |\psi_{01}\rangle = |\psi_{10}\rangle = |00\rangle$ and $|\psi_{11}\rangle = |11\rangle$. It is easily seen that this ensemble can only be used to perform a total bidirectional communication of 1 bit, but it requires 2 bits of communication to create.

However, there are some bidirectional ensembles that can be efficiently prepared, and it is plausible that ensembles of the form (76) can be efficiently prepared. Even though it is not known how to efficiently create these ensembles, it is possible to efficiently create ensembles with the same reduced density matrices (see Appendix B). Because the bidirectional communication that can be performed using the ensemble of Eq. (76) is increased by $2E_U$ via the operation $U$, if it is possible to efficiently create ensembles of this type, it should be possible to apply a communication scheme similar to that used by BHLS such that the average communication is $2E_U$. Such a result would imply that $C^E_\rightarrow = 2E_U$, and therefore $C^E_\rightarrow = 2C^E_\rightarrow = 2E_U$.

VIII. CONCLUSIONS

By considering a quantum superposition of classical messages, we have shown that the capacities for general nonlocal
unitary operations satisfy the inequality $C^E_+=2C_+\leq 2E_U$. For the case of two-qubit unitary operations, we have the further inequalities

$$C^E_+ \leq 2C_+ \leq 2E_U.$$  \hfill (85)

In order to show these inequalities, we have demonstrated that it is possible to put an upper limit on the dimension of the ancilla required for the communication process. This upper limit also simplifies some of the derivations given by BHLS.

We have given an explicit scheme for finding ensembles of states such that the Holevo information may be increased by $E_U$ for a two-qubit unitary operation $U$. Together with the result given by BHLS $C^E_+=\Delta \chi U^\ast$, this implies that $E_U \leq C^E_+$. We have also introduced the concept of bidirectional ensembles, and given upper and lower bounds for the communication that may be performed using these ensembles. For two-qubit unitary operations, we have shown that the bidirectional communication that an ensemble can achieve may be increased by at least $2E_U$: that is, we have the series of inequalities for two-qubit unitary operations

$$C^E_+ \leq 2C_+ \leq 2E_U \leq \Delta \chi U^\ast.$$  \hfill (86)

A promising way of proving equalities is by developing an efficient scheme for remote state preparation of bidirectional ensembles. This problem is a possible direction for future research.

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APPENDIX A: COMMUNICATION RATES

We may express the set of achievable rate pairs without entanglement assistance by

$$S_U=\{(R_+,R_-)|\forall \epsilon>0,\exists t\in \mathbb{Z}^+,\exists n_a\geq tR_-,\exists n_b\geq tR_-,\exists \{V_A^{(i)}\otimes V_B^{(j)}\}_{i=0}^d,\exists d_1,d_2\in \mathbb{Z}^+\ s.t.\ \forall x\in \{0,1\}^{n_a},\forall y\in \{0,1\}^{n_b},F(|y\rangle_{A_1}|x\rangle_{B_1},T_{A_2B_2}\eta_{s_2})_{AB}(|\eta_{s_2}\rangle)|1-\epsilon,,$$

for $|\eta_{s_2}\rangle_{AB}=(V_A^{(i)}\otimes V_B^{(j)}U\ldots

\ldots U(V_A^{(0)}\otimes V_B^{(0)})|x\rangle_{A_1}|y\rangle_{B_1}|0_{d_1}|0_{d_2})$, \hfill (A1)

where $|0_{d_1}\rangle$ and $|0_{d_2}\rangle$ are pure states of ancilla subsystems possessed by Alice and Bob with dimensions $d_1$ and $d_2$, respectively. The set of achievable rate pairs with entanglement assistance is defined by

$$S^E_U=\{(R_+,R_-)|\forall \epsilon>0,\exists t\in \mathbb{Z}^+,\exists n_a\geq tR_-,\exists n_b\geq tR_-,\exists \{V_A^{(i)}\otimes V_B^{(j)}\}_{i=0}^d,\exists d_1,d_2,d_3\in \mathbb{Z}^+\ s.t.\ \forall x\in \{0,1\}^{n_a},\forall y\in \{0,1\}^{n_b},F(|y\rangle_{A_1}|x\rangle_{B_1},T_{A_2B_2}|\eta_{s_2}\rangle_{AB}(|\eta_{s_2}\rangle)|1-\epsilon,$$

for $|\eta_{s_2}\rangle_{AB}=(V_A^{(i)}\otimes V_B^{(j)}U\ldots

\ldots U(V_A^{(0)}\otimes V_B^{(0)})|x\rangle_{A_1}|y\rangle_{B_1}|0_{d_1}|0_{d_2}|\Phi_{d_3})$, \hfill (A2)

where $|\Phi_{d_3}\rangle$ is a maximally entangled state with each subsystem of dimension $d_3$.

APPENDIX B: ENSEMBLE CREATION

An ensemble for which the reduced density matrices possessed by Alice and Bob are the same as for the ensemble of Eq. (76) is given by

$${\mathcal{E}}=\{p_i,q_i,\sigma_i^{(2)}\sigma_j^{(2)}Tr_B(|\phi\rangle)\sigma_j^{(2)}\sigma_i^{(2)}\otimes \sigma_i^{(3)}\sigma_j^{(3)}Tr_A(|\phi\rangle)\sigma_j^{(3)}\sigma_i^{(3)}\},$$  \hfill (B1)

where $p_i=q_i=1/4$.

In order to efficiently prepare this ensemble, Alice and Bob efficiently prepare the two unidirectional ensembles $\{p_i,\sigma_i^{(3)}Tr_A(|\phi\rangle)\sigma_i^{(3)}\}$ and $\{q_j,\sigma_j^{(3)}Tr_B(|\phi\rangle)\sigma_j^{(3)}\}$. Here $i$ is the index chosen by Alice and $j$ is the index chosen by Bob. These two unidirectional ensembles may be treated as one bidirectional ensemble

$$\{p_i,q_j,\sigma_j^{(2)}Tr_B(|\phi\rangle)\sigma_j^{(2)}\sigma_i^{(3)}Tr_A(|\phi\rangle)\sigma_i^{(3)}\}. \hfill (B2)$$

In order to obtain the ensemble given in Eq. (B1), Alice performs the local operation $\sigma_i^{(2)}$, and Bob performs the local operation $\sigma_j^{(3)}$.