TRUNCATIONS OF L-FUNCTIONS IN RESIDUE CLASSES

IGOR E. SHPARLINSKI*

Department of Computing, Macquarie University, Sydney, NSW 2109, Australia
e-mail: igor@ics.mq.edu.au

(Received 9 December, 2005; accepted 10 February, 2006)

Abstract. Let $\chi(n)$ be a quadratic character modulo a prime $p$. For a fixed integer $s \neq 0$, we estimate certain exponential sums with truncated $L$-functions

$$L_{s,p}(n) = \sum_{j=1}^{n} \frac{\chi(j)}{j^s} \quad (n = 1, 2, \ldots).$$

Our estimate implies certain uniformly of distribution properties of reductions of $L_{s,p}(n)$ in the residue classes modulo $p$.

2000 Mathematics Subject Classification. 11L07, 11M38.

1. Introduction. Let $p$ be an odd prime and let $\chi(n)$ be a quadratic character modulo $p$. For a fixed positive integer $s \neq 0$ we define the truncated $L$-functions

$$L_{s,p}(n) = \sum_{j=1}^{n} \frac{\chi(j)}{j^s}, \quad n = 1, 2, \ldots.$$

Various properties of such sums, especially for $s = 1$, have been considered in the literature, see [2, 5, 8, 9] and references therein.

Here we consider the behaviour of these sums in the residue classes modulo $p$. More precisely, in this paper we obtain nontrivial bounds on exponential sums

$$T_s(a; p, M, N) = \sum_{n=M+1}^{M+N} e_p(aL_{s,p}(n)),$$

where

$$e_p(z) = \exp(2\pi iz/p),$$

and $L_{s,p}(n)$ is computed modulo $p$ for $1 \leq n < p$. Then, in a standard fashion, we obtain a uniformity of distribution result for the sequence of fractional parts $\{L_{s,p}(n)/p\}$, $n = M + 1, \ldots, M + N$.

Here we use an approach which is similar to that of [4] however it also needs some additional arguments.

Hereafter, the implied constants in symbols ‘$O$’ and ‘$\ll$’ may depend on the integer parameter $s$ and the real parameter $\varepsilon$ (we recall that $A \ll B$ is equivalent to $A = O(B)$).

*During the preparation of this paper, the author was supported in part by ARC grant DP0556431.
2. Exponential sums.

**Theorem 1.** Let \( \varepsilon > 0 \) be a fixed real number. Let \( M \) and \( N \) be integers with \( 0 \leq M < M + N < p \) and \( N \geq p^{1/2 + \varepsilon} \). Then, for every fixed integer \( s \geq 1 \), the following bound holds:

\[
\max_{\gcd(a, p) = 1} |T_s(a; p, M, N)| \ll N(\log p)^{-1/2}.
\]

**Proof.** We define \( 0^{-s} \equiv 0 \pmod{p} \); thus, \( i^{-s} \pmod{p} \) is defined for all integer \( i \). Then, for any integer \( k \geq 0 \), we have

\[
T_s(a; p, M, N) = \sum_{n=M+1}^{M+N} e_p(aL_{s,p}(n + k)) + O(k).
\]

Therefore, for any integer \( K \geq 1 \),

\[
T_s(a; p, M, N) = \frac{1}{K} W + O(K),
\]

where

\[
W = \sum_{k=0}^{K-1} \sum_{n=M+1}^{M+N} e_p(aL_{s,p}(n + k))
\]

\[
= \sum_{n=M+1}^{M+N} \sum_{k=0}^{K-1} e_p(aL_{s,p}(n) + a \sum_{i=1}^{k} \chi(n + i)(n + i)^{-s})
\]

\[
= \sum_{n=M+1}^{M+N} e_p(aL_{s,p}(n)) \sum_{k=0}^{K-1} e_p(a \sum_{i=1}^{k} \chi(n + i)(n + i)^{-s}).
\]

Applying the Cauchy inequality, we derive

\[
|W|^2 \leq N \sum_{n=M+1}^{M+N} \sum_{k=0}^{K-1} e_p \left( a \sum_{i=1}^{k} \chi(n + i)(n + i)^{-s} \right)^2.
\]

For each \( K \)-dimensional \( \pm 1 \)-vector \( (\vartheta_1, \ldots, \vartheta_K) \in \{-1, 1\}^K \) we see that for \( 1 \leq n < p - K \),

\[
\frac{1}{2K} \prod_{i=1}^{K} (1 + \vartheta_i \chi(n + i)) = \begin{cases} 1, & \text{if } \chi(n + i) = \vartheta_i, \quad i = 1, \ldots, K, \\ 0, & \text{otherwise,} \end{cases}
\]

Therefore we derive from (2) (estimating the contribution of each of the at most \( K \) possible terms with \( p - K \leq n \leq p \) as \( K^2 \)),

\[
|W|^2 \leq \frac{N}{2K} \sum_{(\vartheta_1, \ldots, \vartheta_K) \in \{-1, 1\}^K} \sum_{n=M+1}^{M+N} \prod_{i=1}^{K} (1 + \vartheta_i \chi(n + i))
\]

\[
\times \left| \sum_{k=0}^{K-1} e_p \left( a \sum_{i=1}^{k} \vartheta_i (n + i)^{-s} \right) \right|^2 + NK^2.
\]
For every vector \((\vartheta_1, \ldots, \vartheta_K) \in \{-1, 1\}^K\), one easily verifies that

\[
\sum_{n=M+1}^{M+N} \prod_{i=1}^{K} (1 + \vartheta_i \chi(n + i)) \sum_{k=0}^{K-1} e_p \left( a \sum_{i=1}^{k} \vartheta_i (n + i)^{-s} \right) \bigg|_{s} \bigg|_{s}^2
\]

\[
= \sum_{0 \leq m, k \leq K-1} \sum_{n=M+1}^{M+N} \prod_{i=1}^{K} (1 + \vartheta_i \chi(n + i)) \times e_p \left( a \sum_{i=1}^{k} \vartheta_i (n + i)^{-s} - a \sum_{i=1}^{m} \vartheta_i (n + i)^{-s} \right).
\]

We observe that each sum over \(n\) splits into at most \(2^K\) sums of the form

\[
\sigma_{\rho, g, f}(M, N) = \rho \sum_{n=M+1}^{M+N} \chi(g(n)) e_p (f(n)),
\]

where \(\rho = \pm 1\), \(g(X) \in \mathbb{Z}[X], f(X) \in \mathbb{Z}(X)\) and \(\deg g, \deg f = O(K)\). We observe that if \(|k - m| \geq 2\) then \(f(X)\) is a nonlinear rational function modulo \(p\), and also for every \(k\) and \(m\), there is only one sums for which the corresponding polynomial \(g(X) = 1\) (otherwise \(g(X)\) has no multiple roots modulo \(p\)). Thus, using the standard reduction between complete and incomplete sums (see \([1]\)) we derive from the Weil bound see \([7, Theorem 3, Chapter 6]\), that

\[
\sigma_{\rho, g, f}(M, N) \ll Kp^{1/2} \log p,
\]

if either \(f\) is a nonlinear rational function modulo \(p\) or \(g\) is a nonconstant squarefree polynomial modulo \(p\). Thus (3) applies for all \(O(2^K N^2)\) sums \(\sigma_{\rho, g, f}(M, N)\), except at most \(O(K)\) such sums (as we have seen, at most one such sum may occur for \(O(K)\) pairs of \(k\) and \(m\) with \(|k - m| \leq 1\)). Estimating the exceptional sums \(\sigma_{\rho, g, f}(M, N)\) trivially as \(\sigma_{\rho, g, f}(M, N) \ll N\), and putting everything together, we obtain

\[
W^2 \ll \frac{N}{2^K} \sum_{(\vartheta_1, \ldots, \vartheta_K) \in \{-1, 1\}^K} (2^K K^3 p^{1/2} \log p + KN) + NK^2
\]

\[
\ll 2^K K^3 Np^{1/2} \log p + KN^2.
\]

Therefore, by (1), we derive

\[
T_3(a; p, M, N) \ll K^{-1/2}N + 2^{K/2} K^{-1/2} N^{1/2} p^{1/4} (\log p)^{1/2} + K.
\]

Taking \(K = [0.5 \varepsilon \log p]\), we finish the proof. \(\Box\)

3. Discrepancy. We recall that the discrepancy \(D\) of a sequence of \(M\) points \((\gamma_j)_{j=1}^M\) of the unit interval \([0, 1]\) is defined as

\[
D = \sup_{\mathcal{I}} \left| \frac{A(\mathcal{I})}{M} - |\mathcal{I}| \right|,
\]

where the supremum is taken over intervals \(\mathcal{I} = [\alpha, \beta] \subseteq [0, 1]\) of length \(|\mathcal{I}| = \beta - \alpha\) and \(A(\mathcal{I})\) is the number of points of this set which belong to \(\mathcal{I}\) (see \([3, 6]\)).
For an integer $a$ with $\gcd(a, p) = 1$, we denote by $D_{s, p}(M, N)$ the discrepancy of the sequence of fractional parts

$$\left\{ \frac{L_{s, p}(n)}{p} \right\}, \quad M + 1 \leq n \leq M + N.$$

Using the Erdős–Turán bound (see [3, 6]), which gives a discrepancy bound in terms of exponential sums, we derive:

**Theorem 2.** Let $\varepsilon > 0$ be a fixed real number. Let $M$ and $N$ be integers with $0 \leq M < M + N < p$ and $N \geq p^{1/2+\varepsilon}$. Then, for every fixed integer $s \geq 1$, the following bound holds:

$$D_{s, p}(M, N) \ll N(\log p)^{-1/2} \log \log p.$$

**References**