Charge fluctuations and Hall effect in dusty plasmas

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In a magnetized dusty plasma, Hall effect can arise due to collisional processes when, e.g., ion-dust collision decouples the ions from the magnetic field. In this case, due to the dust stoppage of the ions, the Hall electric field can be generated over large scales. The same collisional processes are also responsible for dust charge fluctuations. Therefore, the Hall effect and dust charge fluctuations are intricately linked to each other through the same mechanism. The inertial scale over which the Hall effect is dynamically important depends not only on the value of the grain charge times the ratio of the grain to the plasma (ion) number densities but also on the ratio of the ion-dust to the electron-dust collision frequencies. This scale can become arbitrarily large when these collision frequencies are comparable. We show that the charge dynamics, though linked to the Hall scale, does not directly affect the low frequency waves in the medium. Since charge fluctuation modifies the length scale over which Hall operates, the normal mode in such a medium is only indirectly affected although wave suffers weak damping due to collision. © 2009 American Institute of Physics. [DOI: 10.1063/1.3207862]

I. INTRODUCTION

The presence of charged, micron sized dust particles can affect the plasma as well as laboratory environment. For example, it is believed that dust is responsible for formation of spokes in the Saturn’s ring. The formation of stars and planets are facilitated by the condensation of the dust grain component. Therefore, space and astrophysical plasmas such as those in the cometary tails, molecular clouds, and planetary nebulae, etc., are in general complex dusty plasmas.\textsuperscript{1–5} In the laboratory environment, interaction of plasmas with chamber walls can affect efficiency and lifetime of various devices.\textsuperscript{6} For example, the characteristics of Hall thruster, also known as closed-drift thruster, crucially depend on the interaction of the plasma with the wall impurities.\textsuperscript{7,8} In fusion devices, impact of dust on the plasma near first wall is not well understood. It is believed, however, that dynamics of dust particles near the wall may significantly modify plasma properties.\textsuperscript{9–13}

In a number of electronic devices, e.g., thermal emission converters, plasma diodes and plasma surface deposition and etching devices, in which plasma is confined by the magnetic field of various geometries, the presence of natural contamination or dust modifies plasma properties. For example, the dust contamination of devices with pulsed external current circuits (e.g., thermal emission converters and plasma diodes) can induce the temperature-gradient driven flows of ions.\textsuperscript{14} The charged dust in the plasma sheath often provides a diagnostic tool of the sheath characteristics.\textsuperscript{6,15,16} Thus, dust is an important constituent of many space and laboratory plasmas. Owing to the complexities of the dusty medium, by complex dusty plasmas, we shall imply here a three-component plasma consisting of electrons, ions, and charged grains.\textsuperscript{17,18} The role of neutral particles will be completely neglected. It is known that even in such a simple model, novel spatial and temporal scales appear in the system.\textsuperscript{6,19–24}

Although charging of the grain could be due to several competing physical processes occurring simultaneously, inelastic collision of the plasma particles with the grain is one of the most important charging mechanisms. The dust surface acts as a physical, thermal, and momentum sink for the plasma particles.\textsuperscript{6,10,19,21,22} For example, if the plasma particle sticks to the dust surface, it cannot carry the momentum because it is attached to an essentially immobile dust. Therefore, plasma charge and current density may plummet in the presence of dust particle. The ambient physical conditions determine the details of plasma sticking and ensuing recombination on the grain surface. As an example, in the planetary rings, grains may carry more than a thousand electronic charge,\textsuperscript{21,24} whereas in the dark molecular clouds, which is very cold (\(\sim 10–100\) K), grains may have \(\pm 2, \pm 1\), and 0 charges.\textsuperscript{3,5} Therefore, the grain charging could be modeled either as stochastic or continuous process. In the present work, we shall model the grain charging in the framework of a continuum model.

The dust particles are generally much heavier than the plasma particles and this large difference offers for considerable simplification and the dynamics of a dusty plasma can be meaningfully investigated in either of the two limits: (i) the dust particles in the plasma can be assumed micron sized and immobile. Therefore, dust will provide a stationary background for the perturbations propagating in the plasma, or, (ii) the perturbations are of the order of or less than the typical plasma frequencies of the dusty fluid (of order a few hertz for micrometer-scale dust grains) and wavelengths are visible to the bare eyes. As the purpose of the present paper is to demonstrate the interrelationship between the charge...
fluctuation and the Hall effect, we shall keep the model simple and assume an immobile dusty background.\textsuperscript{17,18,20}

The collisional, magnetized dusty medium is inherently dispersive. In an immobile dusty background, in the absence of collisions, the motion of the lighter plasma component will cause a charge separation between plasma particles and dust grains. This will invariably lead to the generation of an electric field. However, in the presence of collisions, such a field dominates the dissipative effects like Ohm diffusion only if the electron and ion cyclotron frequencies exceed the electron-dust and ion-dust collision frequencies, respectively.\textsuperscript{25,26} Otherwise, resistive effects may smear the Hall electric field completely. We shall assume that electrons and ions are well magnetized and hence ignore the resistive effect on the magnetic field evolution. This implies that the fluid is frozen in the magnetic field and the Hall electric field is caused by the relative drift of the plasma particles with respect to the immobile dust. Physically, in the frame of stationary dust, this means that the relative drift between plasma particles due to collisions is very small compared with the transverse gyration of the particles across the magnetic field resulting locally in the plasma particles going away or coming close to a stationary observer in the dust frame resulting in a time-dependent Hall electric field. Thus Hall field is generated over the plasma-cyclotron time scale and, depending on the sign of the grain charge the Hall scale, can become arbitrary large.\textsuperscript{20} Therefore, in many space environments, the Hall MHD could be the only proper description of the dusty plasma dynamics.\textsuperscript{20,27,28}

The collisions between plasma particles and charged grains are responsible for some novel features in the dusty medium. The new collective behavior is known to exist in such a plasma due to the dust charge fluctuations, an offshoot of collision.\textsuperscript{19,20} In addition, collisional processes can strongly influence the nonlinear wave modes in a dusty medium.\textsuperscript{25,26,30} Indeed, plasma-dust collisions play an important role in exciting parametric instability.\textsuperscript{27,28} Thus, it is desirable to investigate the dynamics of such a plasma in the presence of Hall effect and charge fluctuation simultaneously. We should also investigate the normal mode in such a magnetized, dispersive, charge fluctuating dusty medium.

Therefore, the primary motivation of this work is to examine the inter-relationship between the charge fluctuation and the Hall effect in a magnetized, collisional dusty medium. It is shown that the dust charge fluctuations can significantly modify the Hall scale and thus the inertial scale over which Hall effect operates appears as a function of the charge fluctuations. Furthermore, we investigate the effect of the charge fluctuations on the wave properties of the medium and show that it has no direct role in the normal mode behavior of the medium. The basic set of equations is discussed in Sec. II. It is shown that although the dust particles provide the stationary background, when the electrons and ions are magnetized, the relative drift between the charged grains and the plasma particles (electrons and ions) gives rise to the Hall diffusion in the medium. The dependence of the Hall scale on the collision frequencies and the dust charge suggests that the Hall term in the induction equation can become comparable to a representative inhomogeneity scale in the system. In the low-frequency limit, the multifluid description of the collisional dusty plasma can be reduced to the single fluid description with the Hall effect. In Sec. III, the normal mode behavior in such a plasma is investigated. In Sec. IV, the discussion and summary of the work is given.

II. BASIC MODEL

We start with the three-component description of a cold dusty plasma and assume that the dust grains provide a stationary background. The dynamics of such a plasma is given in terms of the continuity and momentum equations for respective species. The continuity equation is given by

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i v_i) = -n_i \sigma_i \frac{\partial E_{\parallel}}{\partial t},$$

(1)

Here, $n_i$ is the mass density, $v_i$ is the velocity, and $j$ stands for electrons and ions. The collision term on the right hand side is the moment of the Krooks operator.\textsuperscript{31} The plasma density $n_i$ relaxes to $\rho_{ij}$ over the collisional time. The momentum equations are

$$0 = -en_i \left( E + \frac{v_i \times B}{c} \right) - \rho_i v_i, \quad (2)$$

$$\rho_i \frac{dv_i}{dt} = en_i \left( E + \frac{v_i \times B}{c} \right) - \rho_i v_i, \quad (3)$$

The electron inertia has been neglected while writing Eq. (2).

In Eqs. (2) and (3), we have the Lorentz force term with $E$ and $B$ as the electric and magnetic fields, respectively, $e$ is the electric charge, $Z$ is the number of charge on the grain, $c$ is the speed of light, $n_i$ is the number density, and $\nu_{id}$ is the ion-dust collision frequency of the dust with $j$th species. The electron-dust and ion-dust collision rates for the negatively charged grains are given by\textsuperscript{32}

$$\langle \sigma v \rangle_{id} = \pi a^2 S_i \left( \frac{8T_i}{m_e} \right)^{0.5} \left[ 1 + \frac{1}{4\tau + 3Z} \right]^{0.5} \times \exp \left[ - \frac{Z^{1.5}}{\tau (1 + Z^{0.5})} \right],$$

$$\langle \sigma v \rangle_{ed} = \pi a^2 S_e \left( \frac{8T_e}{m_i} \right)^{0.5} \left[ 1 + \frac{|Z|}{\tau} \right] \left[ 1 + \sqrt{\frac{2}{\tau + 2|Z|}} \right].$$

(4)

While writing Eq. (4), we assumed that the grains are spherical with radius $a$. Here, $S_i$ is the sticking coefficients of the $j$th plasma particle with the dust and $\tau = aT/e^2$ with $T = T_i = T$. For the micron-sized grains carrying $Z = 10$ electron charge, the ion-dust collision rate in a 1 eV plasma with the sticking coefficient $S_i = 1$ and $S_e = 0.5$ is

$$\langle \sigma v \rangle_{id} = 4 \times 10^{5} T_i^{1/2} d^{2}, \quad (5)$$

where $T_i$ is the gas temperature and $a$ is the grain radius in the units of $10^4$ K and $10^{-5}$ cm, respectively. The electron-grain collision rate for the micron-sized grain is
\[ \langle \sigma v \rangle_{cd} = 4 \times 10^7 T_8^{1/2} a_5^2, \]  

which is much smaller than the ion-dust collision rate. It is clear from Eq. (4) that the electron-dust collision rate drops almost to zero when grains acquire 100 or more negative charge. Clearly, it is difficult to imagine grains acquiring $10^4$ charge in this model as the probability of an electron colliding with the grains is close to zero for $Z \sim 100$. Therefore, the above collisional model is limited and cannot describe grains carrying a large number of electron charges.

Note that the plasma-dust collisions are not only responsible for the dust charging but (depending upon the electron affinity of the dust material and the ambient physical conditions) they can as well knock off the electrons from the dust surface. Therefore, in general, we should have an additional term in Eqs. (1)–(3) representing the collisional discharging of grains. However, since collisional detachment from the dust surfaces requires higher pressure and temperature, its role in the present cold dusty plasma model has been neglected.

We define $\rho = \rho_i + \rho_e \approx \rho_i$, $\rho_0 = \rho_{0e} + \rho_{0i} \approx \rho_{0e}$, and $\mathbf{v} = (\rho_i \mathbf{v}_i + \rho_e \mathbf{v}_e)/\rho \approx \mathbf{v}_i$. The continuity and the momentum equations for the bulk fluid are

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = - \nu_{id} (\rho - \rho_0),
\]

\[
\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{E} + \frac{\mathbf{J} \times \mathbf{B}}{\epsilon} - \rho \nu_{ed} \mathbf{v},
\]

where use has been made of the plasma quasineutrality condition $n_i + Z n_d = n_e$ while writing Eq. (8). The current density $\mathbf{J} = - e n_e \mathbf{v}_e + e n_i \mathbf{v}_i$. We note that, while deriving continuity and momentum equations, we assumed the negligible electron inertia and thus the dust-electron collisions do not contribute significantly to the momentum of the bulk fluid.

In order to derive the dust charge fluctuation equation, we use the charge conservation equation

\[
\frac{\partial \rho_i}{\partial t} + \nabla \cdot \mathbf{J} = 0,
\]

where $\rho_i = - e n_e + e n_i + Z n_d$ is the charge density. Making use of Eq. (1) in the above equation, we obtain following charge fluctuation equation:

\[
\frac{dZ}{dt} = \frac{\nu_{id} (n_i - n_{0i}) - \nu_{ed} (n_e - n_{0e})}{n_d}.
\]

We note that the grain charge relaxes to its equilibrium value $Z_0$ over the collisional time, only if the ambipolar condition, $\nu_{id} n_{0i} = \nu_{ed} n_{0e}$, is satisfied around the Debye sphere of the grain. The ambipolar condition around a spherical grain surface is similar to the condition in a plasma sheath, which forms in bounded plasmas in order to prevent the runaway electric field at the wall.

Making use of $\nu_{id} = n_d \langle \sigma v \rangle_{id}$ and quasineutrality condition, the charge fluctuation equation can be recast as

\[
\frac{dZ}{dt} + \nu_{ed} Z = \alpha \rho,
\]

where $m_i \alpha = \langle \sigma v \rangle_{id} - \langle \sigma v \rangle_{ed}$ is the difference between the electron-dust and ion-dust collision rates. While writing Eq. (11), we have assumed $\nu_{id} n_{0i} = \nu_{ed} n_{0e}$. Equation (11) connects the dust charge equation to the plasma mass density. The dust charge Eq. (11) can be rewritten in the familiar form

\[
\frac{dQ}{dt} = I_e + I_i,
\]

if we write the electron and ion currents as

\[
I_e = - e n_e \langle \sigma v \rangle_{id},
\]

\[
I_i = e n_i \langle \sigma v \rangle_{id}.
\]

Then writing $I_e + I_i = e \alpha \rho - Z e v_{ed}$ we recover Eq. (11) from Eq. (12).

It is clear from Eq. (11) that when $\nu_{id} = \nu_{ed}$ i.e., $\alpha = 0$, the charge on the dust decays exponentially as $Z = Z_0 e^{-t/\nu_{ed}}$. Thus, the ambipolar condition under which Eq. (11) has been derived reduces to $n_{0i} = n_{0e}$ implying $Z_0 = 0$. Therefore, if the electrons and ions collide to the dust at equal rates at some initial time $t = 0$, the dust will not carry any charge as instantaneous recombination will destroy any residual charge on the grain surface. This is consistent with the fact that grain charging occurs precisely because of the difference between collision rates of the plasma particles. We note that the form of Eq. (11) will remain unchanged in both unmagnetized and magnetized cases since it has been derived from the general, charge conservation equation. The presence of a magnetic field will affect only the collision frequencies in the transverse (to the ambient field) direction. Since the purpose of this work is to explore the inter-relationship between charge fluctuation and Hall effect, role of the magnetic field on modifying the plasma-dust collision rates will have no direct bearing on the present investigation.

As noted above, in an immobile dusty background, the motion of the lighter plasma component causes a charge separation between plasma particles and dust grains. This invariably leads to the generation of an electric field. Defining $\omega_J = eB/m_e$ as the cyclotron frequency of the plasma particle, we can quantify the level of the plasma magnetization by defining the Hall parameter, $\beta_J = \omega_J / v_J$, which measures the relative strength of the Lorentz force term against the collisional momentum exchange term. In $\beta_J \gg 1$ limit, i.e., when the electrons and ions are strongly magnetized, writing

\[
v_e = \left(1 - \frac{Z n_d}{n_e}\right) v - \frac{\mathbf{J} \times \mathbf{B}}{en_e},
\]

from the electron momentum equation, $c \mathbf{E} + v_e \times \mathbf{B} = 0$, we get following induction equation:\n
\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[\left(1 - \frac{Z n_d}{n_e}\right) \mathbf{v} \times \mathbf{B} - \left(\frac{\mathbf{J} \times \mathbf{B}}{en_e}\right)\right].
\]

Note that unlike ideal MHD, where relative drift between the electrons and ions is responsible for the Hall effect, the pres-
ence of immobile dust is responsible for the current and ensuing Hall effect. Indeed, even when relative drift between the electrons and ions is zero, a nonzero current \( J = -Zn_{ed} \rho \) may still exist due to the presence of charged dust. Therefore, the Hall effect in a dusty plasma can in general appear either due to the relative drift between plasma particles or due to the presence of charged grains.\(^{20}\)

It is clear from the induction equation that the characteristic convective and Hall time scales are

\[
 t_{\text{conv}} = \frac{L}{(1 - \frac{Zn_{ed}}{n_e}) v_A}, \quad t_{\text{Hall}} = \left( 1 + \frac{Zn_{ed}}{n_i} \right) \left( \frac{L}{\delta_i} \right) t_{\text{Alf}},
\]

where \( L \) is the typical length of the inhomogeneity of the system, \( \delta_i = v_A / \omega_i \) is the ion skin depth where \( v_A = B / \sqrt{4 \pi \rho} \) is the Alfvén speed and \( t_{\text{Alf}} = L / v_A \) is the Alfvén crossing time. Since, from Eq. (11),

\[
 \frac{Zn_{ed}}{n_i} \sim \frac{1 - v_{ed} / v_{ed}}{1 + \omega / v_{ed}} \sim \left( 1 - \frac{v_{id}}{v_{ed}} \right),
\]

the Hall time can be written as

\[
 t_{\text{Hall}} \sim \left( 1 - \frac{v_{ed}}{v_{ed}} \right) \left( \frac{L}{\delta_i} \right) t_{\text{Alf}}.
\]

Clearly, temporal scale of the Hall effect depends on the grain charge fluctuations. Equation (4) suggests that, at some initial time \( t=0 \), the electron-dust collision rate is larger than the ion-dust collision rate and thus the electrons stick to the grains at a faster rate than the ions. This results in the local imbalance of the plasma particles. Therefore, there are two processes occurring near an immobile dust grain: the plasma oscillates to offset the local charge imbalance and the gyration of plasma particles against the magnetic field. Thus, the time over which the electric field generation against the fixed dust background takes place is proportional to the ratio of the plasma and cyclotron frequencies and inversely proportional to the quasineutrality factor\(^{20}\) \((1 - Zn_{ed} / n_e)\). With the grain becoming more and more negative, the electron-dust collisions start to drop exponentially and the ion-dust collisions start to increase. However, when both these frequency match each other, the grain relaxes to zero equilibrium value and thus \( J \rightarrow 0 \) and the Hall effect disappears. Therefore, when both plasma collision frequencies exactly match, i.e., \( v_{id} = v_{ed} \), the Hall term in the above expression, Eq. (18), disappears and the fluid, instead of being frozen in the field and plasma vortex,\(^{20}\) becomes frozen only in the field.

In the ideal MHD, the Hall effect appears in Eq. (15) as an additional term proportional to the ion-inertial length \( \delta_i \), which implies that if the dynamical scale is small enough, this effect becomes important. The Hall effect is therefore often ignored while studying large-scale macroscopic properties of the fluid. In a dusty plasma, however, this inertial scale is

\[
 \delta_{\text{Hall}} = \frac{\delta_i}{1 - \frac{Zn_{ed}}{n_e} \left( 1 - \frac{v_{id}}{v_{ed}} \right)}. \quad (19)
\]

Therefore, the Hall effect operates on the modified ion-inertial scale and depends on the grain charge fluctuation. The scale over which the Hall effect operates could become large when \( v_{id} \rightarrow v_{ed} \). However, since \( v_{ed} \rightarrow v_{ed} \), it also implies that \( J \rightarrow 0 \), and the Hall field \( \sim J \times B \) becomes very weak. Therefore, there is a tradeoff between Hall scale and the strength of the Hall field in the \( v_{id} \rightarrow v_{ed} \) limit. We can conclude that the grain charge fluctuations and the Hall effect are inherently linked to each other in a dusty plasma. In the absence of charge fluctuations, \( \delta_{\text{Hall}} \) reduces to the known expression\(^{20}\).

The set of Eqs. (7), (8), and (15) along with Maxwell’s equation,

\[
 \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad (20)
\]

forms a complete set. We shall utilize these equations to investigate the wave properties in a charge fluctuating magnetized medium.

### III. WAVES IN THE MEDIUM

We investigate wave properties of the medium by assuming that dust provides the fixed background. The role of dust on the normal mode behavior of the medium is reflected through the collisions with the plasma particles. Although the charge dynamics Eq. (11) is part of the basic set of equations, it plays no role except indirectly affecting the inertial scale over which Hall effect operates. A homogeneous background with no flow \( (\mathbf{v} = 0) \) and \( E = 0 \) is assumed. The linearized Eqs. (7), (8), (15), and (20) can be written as

\[
 \frac{\partial \mathbf{E}}{\partial t} + \nabla \cdot \left( \mathbf{J} \times \mathbf{B} \right) = -\mathbf{J} \times \mathbf{B} + \mathbf{J} \times \mathbf{B} = -Zn_{ed} \rho \mathbf{E} - \frac{\mathbf{J} \times \mathbf{B}}{c} - \rho \nabla v_{id} \nabla \mathbf{E}, \quad (21)
\]

\[
 \rho \frac{\partial \mathbf{B}}{\partial t} = -Zn_{ed} \nabla \mathbf{E} + \frac{\mathbf{J} \times \mathbf{B}}{c} - \rho \nabla v_{id} \nabla \mathbf{E}, \quad (22)
\]

\[
 \frac{\partial \mathbf{B}}{\partial t} = R \nabla \times (\mathbf{J} \times \mathbf{B}) - \mathbf{E} + \nabla \times \frac{\mathbf{J} \times \mathbf{B}}{en_e}, \quad (23)
\]

\[
 \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad (24)
\]

where \( R = (1 - Zn_{ed} / n_e) \), and

\[
 \nabla \times \mathbf{E} = \frac{4 \pi}{c} \frac{\partial \mathbf{J}}{\partial t}. \quad (25)
\]

The linearized quantities \( \delta f \) are proportional to \( \exp(i\omega t + ik \cdot \mathbf{x}) \) where \( \omega \) is angular frequency and \( \mathbf{k} \) is the wave vector. Defining \( \tilde{\omega} = (\omega - i n_{ed}) \), Eqs. (21) and (22) can be written as

\[
 \tilde{\omega} \rho \mathbf{E} + \rho \mathbf{E} \cdot \mathbf{k} = 0, \quad (26)
\]
\[ i\bar{\omega} \hat{\nu} = -\frac{Ze_n d}{\rho} \hat{\mu} + \frac{i[k \cdot \hat{B} ] \cdot \Delta B - (B \cdot \Delta B) k}{4\pi \rho}. \]  

(27)

Since focus of the present work is low frequency fluctuations, it is assumed that the plasma quasineutrality is maintained to the linear order, i.e., \( \delta n_i + n_d \delta Z = \delta n_e \), implying \( k \cdot \Delta \mathbf{E} = 0 \). Hence charge fluctuation does not enter the linearized Eqs. (21)–(25) directly. In order to see why charge dynamics gets decoupled from the bulk fluid dynamics in the low frequency limit, we see from Eq. (11) that \( \omega = v_{id} \), implying \( \omega \ll v_{ed} \), thus, the bulk fluid momentum is not modified due to the charge fluctuations. What if there is a charge build up and \( k \cdot \Delta \mathbf{E} \neq 0 \)? In such an event, the typical time scale over which plasma responds to offset this charge build up occurs over electron and ion plasma time scales, thus, high frequency response of the medium needs to be considered. The electron and ion inertia will become important in this case and present single fluid formulation is no longer valid since collision will not be able to glue the medium together. Rather, it will cause the dissipation of the high frequency waves.

Dotting Eq. (27) with \( k \), we obtain

\[ \bar{\omega} k \cdot \hat{\mu} = \frac{k^2}{4\pi \rho} (B \cdot \Delta B), \]  

(28)

Making use of \( k^2 \Delta \mathbf{E} = -(\omega/c) k \times \Delta \mathbf{B} \), Eq. (27) can be written as

\[ \bar{\omega} \hat{\nu} = \frac{[(k \cdot B) \Delta B - (B \cdot \Delta B) k]}{4\pi \rho} - \frac{Ze_n d}{\rho} \frac{\omega}{c k^2} (k \times \Delta B). \]  

(29)

The induction Eq. (23) is given by

\[ i\omega \Delta B = iR[(k \cdot B) \hat{\mu} - (\hat{\nu} \cdot \Delta B) B] + \frac{\omega^2 \mu k}{\omega_i} \Delta B. \]  

(30)

Here, \( \hat{\nu} = k/|k|, \hat{B} = B/|B|, \hat{k} \cdot \hat{B} = \cos \theta = \mu, \omega_i^2 = k^2 V_i^2 \), and \( V_i^2 = B^2/(4\pi \rho) \). Making use of Eqs. (28) and (29), the induction Eq. (30) can be written as

\[ i\omega \Delta B = \frac{\omega^2 \mu k}{\omega} \Delta B + \frac{\omega^2}{\omega_i} \mu k \times \Delta B. \]  

(31)

Here, \( F = Zn_d / n_e \). We note that in the absence of collisions, Eq. (31) is identical to Eq. (27) of Ref. 20. We obtain the following dispersion relation from Eq. (31):

\[ (\omega^2 - R \mu^2 \omega_0^2) \Delta B = R \omega_0^2 \hat{B} \cdot \Delta B - \hat{k} \mu \omega \]  

\[ + i\omega \omega_e \mu \left( RF - \frac{\omega^2}{\omega_i} \right) (k \times \Delta B). \]  

(32)

In the absence of collisions, when the wavevector is aligned to the ambient magnetic field, i.e., \( \mu = 1 \), Eq. (32) reduces to the form

\[ \omega^2 = R \omega_0^2 \pm \omega \omega_e \left( RF - \frac{\omega^2}{\omega_i} \right), \]  

(33)

which is identical to Eq. (29) of Ref. 20. Note that in the absence of dust, i.e., when \( R = 1 \) and \( F = 0 \), Eq. (33) describes the Alfvén, whistler, and ion-cyclotron modes, which are normal modes in a two-component Hall MHD. In a dusty plasma, these modes are considerably modified. For example, in the low frequency limit, \( \omega^2 \ll \omega_i^2 \), ignoring the term \( \omega_i^2 \), dispersion relation Eq. (33) describes the bulk-modified low frequency ion-cyclotron wave \( \omega = R \omega_0 \) (when \( \omega_i^2 / \omega_0^2 \gg RF \)) and high frequency whistler \( \omega = \omega_i^2 (F \omega_i) \) modes. The condition for the whistler excitation, i.e., the ambient physical conditions under which the fluctuations at \( \omega_i^2 / \omega_0^2 \ll RF \) can exist in the medium, can be restated in terms of the ion-skin depth and the wavelength of fluctuations as \( \delta_i \leq \lambda \) since \( R \in [0, 1] \) and \( F \in [0, 1] \). Recall that since the Hall effect is the cause of circularly polarized whistler modes, the condition \( \delta_i \leq \lambda \) in an electron-ion plasma implies that the spatial extent of the Hall field is much smaller than the wavelength of the fluctuations. Therefore, the Hall effect ceases to operate on wavelengths longer than the ion-skin depth and propagation of the whistler waves in such a plasma is not possible. However, in a dusty plasma medium, since the Hall effect operates at modified ion-skin depth, Eq. (19), whistlers with the fluctuation wavelength \( \lambda \approx \delta_i \) can be easily excited in the medium.

Clearly, the wave propagation threshold in a negatively charged dusty medium could be higher than in a fully ionized two-component electron-ion plasma.

In the presence of collisions, we first investigate when the fluctuations are parallel to the background field, i.e., \( \Delta B \parallel B \). Then the dispersion relation is given by

\[ \omega^2 - i v_{id} \omega - R \omega_0^2 = 0. \]  

(34)

One readily recognizes the damped Alfvén mode with \( \omega_i^2 = R \omega_0^2 - v_{id}^2 / 2 \) and \( \omega = \omega_i + i \omega_0 \). Therefore, Alfvén waves can disappear in a collisional dusty medium either due to the severe collisional damping (the damping rate \( \sim v_{id} \)) or when \( R \) becomes very small.
When the magnetic perturbation is perpendicular to the ambient magnetic field, $\mathbf{A} \perp \mathbf{B}$, the dispersion relation becomes
\[
\omega = \mu \omega^2 + (\omega - i \nu_{id}) = R \mu^2 \omega^2 + RF \mu \omega_{id}.
\] (35)

Equation (35) describes a mixture of damped Alfvén and ion-cyclotron modes. Furthermore, when $\mu = 0$, i.e., $k \perp B$, we have either $\omega = 0$ or $\omega = i \nu_{id}$, i.e., a purely imaginary value indicating damping of the oscillations. In the $\omega^2 \ll \omega_{id}^2$ limit, the dispersion relation (35) gives the damped ion-cyclotron mode
\[
\omega = R \mu \omega_{id} + i \nu_{id},
\] (36)

with the damping rate $\omega_{id}$. The threshold of the modified ion-cyclotron mode is controlled by the orientation of its wavevector with respect to the ambient magnetic field. Since $\mu \in [0, 1]$, the maximum threshold is when $\mu = 1$. The excitation of the ion-cyclotron mode in a two-component plasma is related to the finite ion Larmor radius effect, i.e., to the electric field over the ion gyroradius. In three component dusty plasma such a wave is excited because of the presence of the Hall electric field, which is generated when the ion moves against the fixed dusty background. The Hall field is locally caused due to the quasineutrality constraint. Since we are considering low frequency fluctuations in the present case, collisions play the role of gluing the medium together and simultaneously causing the charge to fluctuate. This dual role of collisions is reflected in the modified ion-inertial scale. Unlike the ion-inertial scale of a two-component plasma, it is the combination of the Debye scaled local field and gyroradius over which dusty electrostatic ion-cyclotron mode operates. The role of such an electric field in rotating plasma is well known.\textsuperscript{20,34}

The damped electromagnetic whistler dispersion relation becomes
\[
\omega = \mu \omega^2 F \omega_{ci} - i \nu_{id} \frac{\omega^2}{RF \omega_{id}^2},
\] (37)

with the damping rate $\sim \nu_{id}$. When $\omega^2 \ll \omega_{id}^2$, we recover the dispersion relation for the damped Alfvén wave.

The normal mode investigation of the dusty medium in various limit shows that although the behavior is similar to the usual two-component electron-ion plasma, the temporal and spatial scales over which the modes can be excited are quite different. This is consistent with the physical picture that massive dust introduces new scales in the system. The charge fluctuations only indirectly affect the wave behavior of the medium. This is due to the fact that, for very low frequency waves, the plasma maintains overall quasineutrality and any effect of the charge fluctuating field is completely smeared on the normal mode of the system.

IV. DISCUSSIONS AND SUMMARY

The Hall MHD description of a fully ionized two-component plasma assumes that the relative drift between electrons and ions is nonzero and, therefore, the scale of symmetry breaking with respect to the magnetic field occurs over the ion inertial scale. This scale is generally very small. In a dusty plasma, the Hall effect can be caused by the mere presence of immobile (or mobile) dust grains.\textsuperscript{20} Collisions between plasma particles and charged dust can as well break the symmetry between electrons and ions with respect to the magnetic field and can cause the Hall electric field.\textsuperscript{25} Collisions are also linked to the dust charge fluctuations in a dusty plasma and the dynamics of such a plasma is significantly modified in the presence of charge fluctuations.\textsuperscript{19} It should be expected then that the plasma-dust collisions will cause both the Hall field and charge fluctuation simultaneously.

Therefore, it is not surprising that the ion-inertial scale in dusty plasma is a function of both collisional frequencies and dust charge. The magnetic field plays an important role in grain charging (e.g., the collision frequencies are considerably modified and the charging is not spherically symmetric). However, since the conclusions of the present work are not dependent upon the exact expressions for these frequencies, the results presented here have broader validity.

The charge fluctuations do not directly affect the low frequency wave properties of the dusty medium. This is related to the fact that the dynamical response time of the plasma to the charge fluctuations occurs over fast time scales, whereas the low frequency modes are not directly affected by the charge fluctuations. However, the charge fluctuations considerably modify the Hall scale and thus their effect on the wave properties of the medium is felt via the Hall field operating over scales considerably larger than the ion-skin depth. Since collisions between plasma and dust particles are the primary cause of the grain charge fluctuations, inclusion of the collisions not only makes the three-component plasma moving together as a single bulk fluid but they also cause weak damping of the modes.

The wave properties of the medium are not modified directly in the charge fluctuating medium. Since the modified ion-inertial scale is $\sim 1/(v_{cd} - v_{id})$, whistlers of very large wavelengths can be excited in the medium in the vicinity of $v_{ed} \sim v_{id}$ i.e., when the grain charge approaches a stationary value. This feature is entirely novel to the dusty medium and may have important applications to astrophysical plasmas.

Following is the brief summary of the work.

1. In a magnetized, collisional dusty plasma, the Hall electric field and charge fluctuations are related to each other. The modified ion-inertial scale in such a plasma becomes a function of both dust number densities $|Z| n_{d}/n_e$ and collisional parameters $v_{cd}/v_{id}$.

2. The modified ion-inertial scale can become arbitrary large in a charge fluctuating dusty plasma due to $|Z| n_{d}/n_e \rightarrow 1$ or $v_{cd} \rightarrow v_{id}$ limit. This is due to the fact that symmetry breaking with respect to the magnetic field is caused by the grain inertia, as well as by collisions. Therefore, the Hall scale is a function of both the collision frequencies and the plasma densities.

3. The behavior of normal modes in a dusty medium is significantly modified. Although the physical mechanism of the low frequency normal mode propagation in a dusty medium is similar to the electron-ion plasma, the spatial and temporal scales are very different.
(4) When electron-dust and ion-dust collision frequencies become comparable, the charge on the grain will acquire a stationary value. In this limit, the wavelength of the whistler can become arbitrary large.

(5) The collision plays dual role in the dusty medium. Collision may glue the medium together as a single fluid and thus, undamped low frequency waves can propagate in the medium. However, collision also causes a weak dissipation of the fluctuation energy.

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