DISCREPANCY FOR RANDOMIZED RIEMANN SUMS

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Abstract. Given a finite sequence \( U_N = \{u_1, \ldots, u_N\} \) of points contained in the \( d \)-dimensional unit torus, we consider the \( L^2 \) discrepancy between the integral of a given function and the Riemann sums with respect to translations of \( U_N \). We show that with positive probability, the \( L^2 \) discrepancy of other sequences close to \( U_N \) in a certain sense preserves the order of decay of the discrepancy of \( U_N \). We also study the role of the regularity of the given function.

1. Introduction

Let \( N \in \mathbb{N} \) be a given large number, let \( U_N = \{u_1, \ldots, u_N\} \) be a distribution of \( N \) points in the unit cube \( [-\frac{1}{2}, \frac{1}{2}]^d \), treated as the torus \( \mathbb{T}^d \), and let \( f \) be a real function on \( \mathbb{T}^d \). Suppose that for suitable choices of \( U_N \) and \( f \), the Riemann sums
\[
\frac{1}{N} \sum_{j=1}^{N} f(u_j - x)
\]
are, after an \( L^2 \) average on the variable \( x \in \mathbb{T}^d \), good approximations of the integral
\[
\int_{\mathbb{T}^d} f(s) \, ds.
\]
What corresponding statement can we make concerning those sequences close to the sequence \( U_N \)? Do such sequences mostly share the same good behavior?

2. A randomization argument

In order to start discussing these questions, we introduce the following randomization of \( U_N \); see [3, 6] and also [8, 9]. Let \( d\mu \) denote a probability measure on \( \mathbb{T}^d \). For every \( j = 1, \ldots, N \), let \( d\mu_j \) denote the measure obtained after translating \( d\mu \) by \( u_j \). More precisely, for any integrable function \( g \) on \( \mathbb{T}^d \), we have
\[
\int_{\mathbb{T}^d} g(t) \, d\mu_j = \int_{\mathbb{T}^d} g(t - u_j) \, d\mu.
\]
Let $dt$ denote the Lebesgue measure on $\mathbb{T}^d$. For every sequence $V_N = \{v_1, \ldots, v_N\}$ in $\mathbb{T}^d$ and every function $f \in L^2(\mathbb{T}^d, dt)$, we introduce, for every $t \in \mathbb{T}^d$, the discrepancy

$$D(t, V_N) \overset{\text{def}}{=} \frac{1}{N} \sum_{j=1}^{N} f(v_j - t) - \int_{\mathbb{T}^d} f(s) \, ds.$$ 

Observe that $D(\cdot, V_N)$ is a periodic function with Fourier series

$$\sum_{0 \neq k \in \mathbb{Z}^d} \left( \frac{1}{N} \sum_{j=1}^{N} e^{-2\pi i k \cdot v_j} \right) \hat{f}(k)e^{2\pi i k \cdot t},$$

and the Parseval identity yields

$$D^2(V_N) \overset{\text{def}}{=} \|D(\cdot, V_N)\|^2_{L^2(\mathbb{T}^d, dt)} = \sum_{0 \neq k \in \mathbb{Z}^d} \left| \frac{1}{N} \sum_{j=1}^{N} e^{2\pi i k \cdot v_j} \right|^2 |\hat{f}(k)|^2.$$

We now average $D(V_N)$ in $L^2(\mathbb{T}^d, d\mu_j)$ for every $j = 1, \ldots, N$ and consider

$$\mathcal{D}_{d\mu}(U_N) \overset{\text{def}}{=} \left( \int_{\mathbb{T}^d} \ldots \int_{\mathbb{T}^d} D^2(V_N) \, d\mu_1(v_1) \ldots d\mu_N(v_N) \right)^{1/2}.$$

In this paper we study the relation between $\mathcal{D}_{d\mu}(U_N)$ and $D(U_N)$. In the case $N = M^d$, where $M \in \mathbb{N}$, and

$$U_N = \frac{1}{M} \mathbb{Z}^d \cap \left[ -\frac{1}{2}, \frac{1}{2} \right]^d,$$

the above quantities were studied in relation to the sharpness of a result of Beck [11] and of Montgomery [10] on irregularities of distribution; see Remark 3 below. In [6] two of the authors compared the quantities $D(U_N)$ and $\mathcal{D}_{d\mu}(U_N)$ in the case (1) and when $f$ is the characteristic function of a ball. Here we study the problem in our more general setting, and we are mainly interested in whether the inequality

$$\mathcal{D}_{d\mu}(U_N) \leq c \, D(U_N) \tag{2}$$

holds.

Throughout this paper, the letters $c, C, \ldots$ will denote positive constants, possibly depending on $f$ but independent of $N$, and which may change from one step to the next. On the other hand, different letters $B, \kappa, \ldots$ will denote constants which will not change throughout the paper.
3. An Explicit Formula

We first use a slight modification of an argument in [6] to obtain an explicit formula for $\mathcal{D}_{d\mu}(U_N)$. We have

\begin{equation}
\mathcal{D}_{d\mu}^2(U_N)
= \sum_{0 \neq k \in \mathbb{Z}^d} |\hat{f}(k)|^2 \left( \frac{1}{N} + \frac{1}{N^2} \sum_{j, \ell=1 \atop j \neq \ell}^N e^{2\pi i k \cdot v_j} e^{-2\pi i k \cdot v_{\ell}} d\mu_j(v_j) d\mu_\ell(v_\ell) \right)
\end{equation}

We have

\begin{align*}
\mathcal{D}_{d\mu}^2(U_N) &= \sum_{0 \neq k \in \mathbb{Z}^d} |\hat{f}(k)|^2 \left( \frac{1}{N} + |\hat{\mu}(k)|^2 \left( \frac{1}{N} \sum_{j=1}^N e^{2\pi i k \cdot u_j} \right)^2 - \frac{1}{N} \right) \\
&= \frac{1}{N} \sum_{0 \neq k \in \mathbb{Z}^d} |\hat{f}(k)|^2 (1 - |\hat{\mu}(k)|^2) + \sum_{0 \neq k \in \mathbb{Z}^d} |\hat{f}(k)|^2 |\hat{\mu}(k)|^2 \left( \frac{1}{N} \sum_{j=1}^N e^{2\pi i k \cdot u_j} \right)^2 \\
&= \frac{1}{N} \left( \|f\|_{L^2(\mathbb{T}^d,dt)}^2 - \|f \ast d\mu\|_{L^2(\mathbb{T}^d,dt)}^2 \right) + \|D(\cdot, U_N) \ast d\mu\|_{L^2(\mathbb{T}^d,dt)}^2.
\end{align*}

There are two natural extremal measures. The first one is $d\mu = \delta_0$, the Dirac measure centred at 0. In this case, we have

$$\mathcal{D}_{\delta_0}(U_N) = D(U_N).$$

On the other hand, when $d\mu = dt$, we have

$$\mathcal{D}_{dt}^2(U_N) = \frac{1}{N} \left( \|f\|_{L^2(\mathbb{T}^d,dt)}^2 - \left\| \int_{\mathbb{T}^d} f(t) \, dt \right\|^2 \right),$$

the classical Monte-Carlo error.

Note that if $ND^2(U_N) \geq c$, then $\mathcal{D}_{d\mu}(U_N) \leq c_1 D(U_N)$, and (2) follows easily.

Another very peculiar case is when $D(U_N) = 0$. We observe that in general this does not imply $\mathcal{D}_{d\mu}(U_N) = 0$, so that (2) does not hold. Indeed, let $U_N$ be given by (1). Then

\begin{equation}
\frac{1}{N} \sum_{j=1}^N e^{2\pi i k \cdot u_j} = \begin{cases} 1 & \text{if } k \in M\mathbb{Z}^d, \\ 0 & \text{otherwise}. \end{cases}
\end{equation}
Now choose \( f(t) = \exp(2\pi ik_0 \cdot t) \) for some \( k_0 \in \mathbb{Z}^d \setminus M \mathbb{Z}^d \). Then \( D(U_N) = 0 \). On the other hand, it follows from (3) that

\[
\mathcal{D}^2_{d\mu}(U_N) = \frac{1}{N} (1 - |\hat{\mu}(k_0)|^2) \neq 0
\]

whenever \( |\hat{\mu}(k_0)| \neq 1 \), which is easily fulfilled, particularly by several measures with small support around the origin.

Hence, throughout the paper, we will be interested only in the case when

\[
0 < D(U_N) < N^{−1/2}.
\]

Let \( 0 < \varepsilon_N \leq 1 \). For every probability measure \( d\mu \) supported on the unit cube \([−\frac{1}{2}, \frac{1}{2})^d\), let \( d\mu^{(N)} \) denote the probability measure defined by

\[
\int_{\mathbb{R}^d} g(\xi) d\mu^{(N)}(\xi) = \int_{\mathbb{R}^d} g(\varepsilon_N \xi) d\mu(\xi). \tag{5}
\]

Then \( d\mu^{(N)} \) is supported on the subcube \([−\frac{1}{2}\varepsilon_N, \frac{1}{2}\varepsilon_N)^d\) and can be regarded as a measure on \( \mathbb{T}^d \).

4. Main result

We first state our main result.

**Theorem 1.** Let \( f \in L^2(\mathbb{T}^d, dt) \) and let \( U_N = \{u_1, \ldots, u_N\} \) be a distribution of \( N \) points in the cube \([−\frac{1}{2}, \frac{1}{2})^d\). Assume that \( 0 < D(U_N) < N^{−1/2} \). Let \( d\mu \) be a non-Dirac probability measure on \( \mathbb{T}^d \), let \( d\mu^{(N)} \) be defined by (5) with \( 0 < \varepsilon_N \leq 1 \), and let

\[
\eta_N = \begin{cases} 
\varepsilon_N^{2\alpha} & \text{if } \alpha < 1, \\
\varepsilon_N \log(1 + \varepsilon_N^{-1}) & \text{if } \alpha = 1, \\
\varepsilon_N^2 & \text{if } \alpha > 1.
\end{cases}
\]

(i) If for some \( \alpha > 0 \) and for every \( \rho > 1 \) we have

\[
\sum_{\rho \leq |k| < 2\rho} |\hat{f}(k)|^2 \leq c \rho^{-2\alpha}, \tag{6}
\]

then

\[
\mathcal{D}^2_{d\mu^{(N)}}(U_N) \leq c \eta_N N^{-1} + D^2(U_N). \tag{7}
\]

(ii) If there exists an open cone \( \Omega \subseteq \mathbb{R}^d \) such that for every subcone \( \Gamma \subseteq \Omega \),

\[
\liminf_{\rho \to \infty} \rho^{2\alpha} \sum_{\rho \leq |k| < 2\rho} |\hat{f}(k)|^2 > 0, \tag{8}
\]

then there exist positive constants \( \Delta \leq 1 \) and \( c \) such that if \( \varepsilon_N \leq \Delta \), then

\[
\mathcal{D}^2_{d\mu^{(N)}}(U_N) \geq c \eta_N N^{-1}.
\]

The following corollary shows that, in some sense, good sequences are never alone. Indeed we give conditions on \( \varepsilon_N \) that will ensure that \( \mathcal{D}_{d\mu^{(N)}}(U_N) \) and \( D(U_N) \) are comparable.

**Corollary 2.** Let \( f, U_N \) and \( d\mu \) be as given in Theorem [1].

\[^1\]In this paper every cone starts from the origin.
Let \( f \) be as given in part (i) of Theorem 1 and let

\[
\varepsilon_N \leq \begin{cases} 
(N^{1/2} D(U_N))^{1/\alpha} & \text{if } \alpha < 1, \\
\beta_N & \text{if } \alpha = 1, \\
N^{1/2} D(U_N) & \text{if } \alpha > 1,
\end{cases}
\]

where \( \beta_N \) satisfies

\[
\beta_N^2 \log(1 + \beta_N^{-1}) = ND^2(U_N).
\]

Then \( D_2^2 d\mu(N(U_N)) \leq c D_2^2(U_N) \).

Let \( f \) and \( \Delta \) be as given in part (ii) of Theorem 1 and let \( \kappa > 0 \). Then there exists \( c > 0 \) such that whenever

\[
\Delta \geq \varepsilon_N \geq \begin{cases} 
\kappa(N^{1/2} D(U_N))^{1/\alpha} & \text{if } \alpha < 1, \\
\kappa \beta_N & \text{if } \alpha = 1, \\
\kappa N^{1/2} D(U_N) & \text{if } \alpha > 1,
\end{cases}
\]

we have

\[
D_2^2 d\mu(N(U_N)) \geq c D_2^2(U_N).
\]

Remark 3. Consider the particular case when \( f = \chi_A \), the characteristic function of a convex body \( A \subseteq [-1/2, 1/2]^d \). Then (8) holds with \( \alpha = 1/2 \). Let \( \varepsilon_N = \Delta N D^2(U_N) \). Then

\[
D_2^2 d\mu(N(U_N)) \leq c D_2^2(U_N).
\]

If furthermore the boundary of \( A \) is smooth and has positive Gaussian curvature, then (8) holds with \( \alpha = 1/2 \); see, for instance, [7]. We then have

\[
D_2^2 d\mu(N(U_N)) \geq c D_2^2(U_N).
\]

We recall that if \( A \) is rotated and contracted, then a result of Beck [1] and of Montgomery [10] says that

\[
\int_{\mathbb{R}^d} \int_0^1 \left| \frac{1}{N} \sum_{j=1}^N \chi_{\sigma(rA)(u_j-t)} - \sigma^d|A| \right|^2 \, dt \, d\sigma \geq c N^{-1-1/d}
\]

for every choice of the point set distribution \( U_N \); see also [2, 4, 5]. We also recall that this is not true if the contraction is omitted; see [12, Theorem 3.1].

5. Decay of the Fourier coefficients

The assumption (8) concerns the decay of the Fourier coefficients of \( f \). This behavior can be naturally related to the smoothness of the function \( f \) as follows. Let \( f \in L^2(\mathbb{R}^d) \), define \( \Delta_h f(x) = f(x + h) - f(x) \) and, for every integer \( \ell \geq 1 \), write \( \Delta_h^\ell f = \Delta_h \Delta_h^{\ell-1} f \). Let \( \alpha > 0 \). We say that \( f \) belongs to the Nikol’skii space \( H_2^\alpha(\mathbb{R}^d) \) if there exists \( c > 0 \) such that

\[
\left( \int_{\mathbb{R}^d} |\Delta_h^\ell f(x)|^2 \, dx \right)^{1/2} \leq c |h|^\alpha
\]

for some \( \ell \geq 1 \); see [11, Section 4.3.3].

Proposition 4. Let \( f \in H_2^\alpha(\mathbb{R}^d) \). Then (8) holds.
Proof. Since \( \hat{\Delta_h}f(k) = (e^{2\pi i k \cdot h} - 1) \hat{f}(k) \), we have \( \hat{\Delta_h^\ell}f(k) = (e^{2\pi i k \cdot h} - 1)^\ell \hat{f}(k) \). Let \( h = (1/10, 0, \ldots, 0) \) and \( \Gamma = \{ k \in \mathbb{Z}^d : k_1 > k_2^2 + \ldots + k_d^2 \} \). Observe that when \( k \in \Gamma \) and \( \rho \leq |k| \leq 2\rho \), we have \( |e^{2\pi i k \cdot h} - 1| \geq c \). Therefore
\[
\sum_{k \in \Gamma \atop \rho \leq |k| < 2\rho} |\hat{f}(k)|^2 \leq c \sum_{k \in \Gamma \atop \rho \leq |k| < 2\rho} |(e^{2\pi i k \cdot h} - 1)^\ell \hat{f}(k)|^2 \leq c \sum_{k \in \mathbb{Z}^d} |\hat{\Delta_h^\ell}f(k)|^2
\]
\[
= c \int_{\mathbb{T}^d} |\Delta_h^\ell f(x)|^2 \, dx \leq c |h|^{2\alpha} = c \rho^{-2\alpha}.
\]
Note here that \( h \) is tailored on \( \Gamma \). Since we can cover \( \mathbb{Z}^d \) with a finite number of cones, the proposition follows from the above argument applied to different choices of \( h \). \( \square \)


Lemma 5. Let \( \nu \) be a probability measure supported on \( [-\frac{1}{2}, \frac{1}{2}]^d \). Then either

(i) \( \nu \) is the Dirac measure \( \delta_{t_0} \) at a point \( t_0 \in [-\frac{1}{2}, \frac{1}{2}]^d \), or

(ii) \( 1 - |\hat{\nu}(\xi)|^2 = O(|\xi|^2) \) as \( \xi \to 0 \), and any open cone in \( \mathbb{R}^d \) contains an open subcone \( \Gamma \) such that \( 1 - |\hat{\nu}(\xi)|^2 \geq c|\xi|^2 \) for small \( \xi \in \Gamma \).

Proof. Since \( \nu \) is compactly supported, its Fourier transform \( \hat{\nu} \) is smooth and has Taylor expansion
\[
\hat{\nu}(\xi) = 1 + \nabla \hat{\nu}(0) \cdot \xi + \frac{1}{2} H_{\hat{\nu}}(0) \xi \cdot \xi + o(|\xi|^2),
\]
and so
\[
1 - |\hat{\nu}(\xi)|^2 = 1 - \hat{\nu}(\xi)\hat{\nu}(-\xi) = (\nabla \hat{\nu}(0) \cdot \xi)^2 - H_{\hat{\nu}}(0) \xi \cdot \xi + o(|\xi|^2) = O(|\xi|^2).
\]

Let \( F(\xi) = (\nabla \hat{\nu}(0) \cdot \xi)^2 - H_{\hat{\nu}}(0) \xi \cdot \xi \), and assume that \( F \) does not vanish identically. Let \( \Sigma_{d-1} = \{ \xi \in \mathbb{R}^d : |\xi| = 1 \} \). Since \( F \) is a polynomial, it cannot vanish on an open set, and therefore \( \{ \xi \in \Sigma_{d-1} : F(\xi) = 0 \} \) has empty interior in \( \Sigma_{d-1} \). Since \( F \) is homogeneous and continuous, it follows that for every open cone in \( \mathbb{R}^d \), we can find an open subcone \( \Gamma \) such that \( |F(\xi)| \geq c|\xi|^2 \) for \( \xi \in \Gamma \). Therefore \( 1 - |\hat{\nu}(\xi)|^2 \geq c|\xi|^2 \) for small \( \xi \in \Gamma \).

Assume now that \( F \equiv 0 \). Observe that
\[
\frac{\partial \hat{\nu}}{\partial \xi_j}(0) = -2\pi i \int_{\mathbb{T}^d} x_j \, \nu(x)
\]
and
\[
\frac{\partial^2 \hat{\nu}}{\partial \xi_j \xi_k}(0) = -4\pi^2 \int_{\mathbb{T}^d} x_j x_k \, \nu(x).
\]
Then
\[
\nabla \hat{\nu}(0) \cdot \xi = -2\pi i \int_{\mathbb{T}^d} (x \cdot \xi) \, \nu(x)
\]
and
\[
H_{\hat{\nu}}(0) \xi \cdot \xi = -4\pi^2 \sum_{i,j} \int_{\mathbb{T}^d} \xi_j \xi_k x_j x_k \, \nu(x) = -4\pi^2 \int_{\mathbb{T}^d} (\xi \cdot x)^2 \, \nu(x).
\]
In view of (11) and (12), the inequality (7) follows from (3).

(12)
\[ \|D(\cdot, U_N) \ast d\mu\|_{L^2(T^d, dt)} \leq D(U_N). \]

In view of (11) and (12), the inequality (7) follows from (3).
Let us now prove (ii). By Lemma 5 there exists a subcone $\Gamma \subset \Omega$ such that $1 - |\hat{\nu}(\xi)|^2 \geq m_1|\xi|^2$ for $|\xi| \leq m_2$, $\xi \in \Gamma$. By (3) there exist $m_3$ and $m_4$ such that for $\rho \geq m_3$ we have

$$\sum_{\rho \leq |k| < 2\rho} |\hat{f}(k)|^2 \geq m_4 \rho^{-2\alpha}.$$  

Thus, for $\varepsilon_N < \min\{m_2/4m_1, 1\}$, we have

$$D^2_{d\mu(N)}(U_N) \geq \frac{1}{N} \left( \|f\|^2_{L^2(T^d, dx)} - \|f \ast d\mu(N)\|^2_{L^2(T^d, dt)} \right)$$

$$= \frac{1}{N} \sum_{0 \neq k \in \mathbb{Z}^d} |\hat{f}(k)|^2 (1 - |\hat{\nu}(\xi)|^2)$$

$$\geq \frac{1}{N} \sum_{m_3 \leq 2^j \leq m_2 \varepsilon_N^{-1}} \sum_{k \in \Gamma} |\hat{f}(k)|^2 (1 - |\hat{\nu}(\xi)|^2)$$

$$\geq \varepsilon_N^2 \frac{m_1 m_4}{N} \sum_{m_3 \leq 2^j \leq m_2 \varepsilon_N^{-1}} 2^{-2\alpha} 2^{2j} \geq c \eta N^{-1}.$$  

This completes the proof of Theorem 1.

Remark 6. The estimates from below for $D^2_{d\mu(N)}(U_N)$ contained in Theorem 1 and Corollary 2 depend on suitable estimates for the first term

$$\frac{1}{N} \left( \|f\|^2_{L^2(T^d, dx)} - \|f \ast d\mu(N)\|^2_{L^2(T^d, dt)} \right)$$

in (3). We observe that in our setting the second term may vanish even in rather natural examples. Indeed, let

$$f(x) = \sum_{k \neq 0} \frac{1}{|k|^\gamma} e^{2\pi i k x}$$

for some $\gamma > d/2 + 1$. One can easily check that (3) holds with $\alpha = \gamma - d/2$. Let $U_N$ be as in (1) and $\mu$ be the (normalized) Lebesgue measure restricted to $[-\frac{1}{2}, \frac{1}{2}]^d$, so that, taking $\varepsilon_N = 1/M$, we have

$$\hat{\mu(N)}(k) = N \prod_{j=1}^d \frac{\sin(\pi k_j/M)}{\pi k_j}.$$  

By (1) we have

$$D^2(U_N) = \sum_{k \neq 0} |\hat{f}(Mk)|^2 = \frac{1}{M^{2\gamma}} \sum_{k \neq 0} \frac{1}{|k|^{2\gamma}} = \frac{c_\gamma}{M^{2\gamma}}$$

and

$$\|D(\cdot, U_N) \ast d\mu(N)\|_{L^2(T^d, dt)} = \sum_{k \neq 0} |\hat{f}(Mk)|^2 |\hat{\mu(N)}(Mk)|^2 = 0.$$  

On the other hand observe that, for large $N$,

$$\varepsilon_N = \frac{1}{M} \geq c_\gamma N^{d/2} D(U_N) = c_\gamma M^{d/2 - \gamma},$$  

and therefore we can apply part (ii) of Corollary 2 and obtain the inequality $D^2_{d\mu(N)}(U_N) \geq c D^2(U_N)$. 

7. Conclusion

Let $d\mu^\otimes$ be defined on $(\mathbb{T}^d)^N$ by

$$
\int_{(\mathbb{T}^d)^N} \varphi \, d\mu^\otimes = \int_{\mathbb{T}^d} \ldots \int_{\mathbb{T}^d} \varphi(v_1 - u_1, \ldots, v_N - u_N) \, d\mu^{(N)}(v_1) \ldots d\mu^{(N)}(v_N).
$$

We can now state and prove the result introduced in the abstract.

**Corollary 7.** Let $f$, $U_N$ and $d\mu$ be as given in Corollary 2.

(i) Let $f$ and $\varepsilon_N$ be as given in part (i) of Corollary 2. Then for every $\lambda$ satisfying $0 < \lambda < 1$, there exists a constant $c_\lambda > 0$, independent of $U_N$ and such that $d\mu^\otimes(\{V_N : D(V_N) \leq c_\lambda D(U_N)\}) \geq \lambda$.

(ii) Let $f$, $\Delta$ and $\varepsilon_N$ be as given in part (ii) of Corollary 2. Then for a suitable constant $c > 0$, we have $d\mu^\otimes(\{V_N : D(V_N) > cD(U_N)\}) > 0$.

**Proof.** If (i) holds, then Corollary 2 gives

$$
\int_{\mathbb{T}^d} \ldots \int_{\mathbb{T}^d} D^2(V_N) \, d\mu^\otimes(V_N) \leq c D^2(U_N).
$$

By the Chebyshev inequality, we have

$$
d\mu^\otimes(\{V_N : D(V_N) > c_\lambda D(U_N)\}) \leq \frac{c}{c_\lambda},
$$

and so

$$
d\mu^\otimes(\{V_N : D(V_N) \leq c_\lambda D(U_N)\}) \geq 1 - \frac{c}{c_\lambda}.
$$

A suitable choice of $c_\lambda$ completes the proof of part (i). If (ii) and (iii) hold, then Corollary 2 gives

$$
\int_{\mathbb{T}^d} \ldots \int_{\mathbb{T}^d} D^2(V_N) \, d\mu^\otimes(V_N) \geq c D^2(U_N),
$$

which easily implies $d\mu^\otimes(\{V_N : D(V_N) \geq cD(U_N)\}) > 0$. 

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